$\underbrace{12 + 22 + 6}_{4.2} + \underbrace{6 + 14 + 20 + 4 + 6}_{4.3} + \underbrace{12 + 12}_{4.4} = 114$ Grand total

MATH 241 PS 7

§4.2 / 3b. $f(x,y) = x^2y^2 - x^2y + xy$ $\quad (x_0, y_0) = (-1, 3)$

$f(-1,3) = 9 - 3 - 3 = 3$

$f_x = 2xy^2 - 2xy + y$ so $f_x(-1,3) = -18 + 6 + 3 = -9$

$f_y = 2x^2y - x^2 + x$ so $f_y(-1,3) = 6 - 1 - 1 = 4$

(2) 1st degree: $3 + (-9)(x+1) + (4)(y-3) = \boxed{-15 - 9x + 3y}$

$f_{xx} = 2y^2 - 2y$ so $f_{xx}(-1,3) = 18 - 6 = 12$

$f_{xy} = 4xy - 2x + 1$ so $f_{xy}(-1,3) = -12 + 2 + 1 = -9$

$f_{yy} = 2x^2$ so $f_{yy}(-1,3) = 2$

2nd degree (4)

$$\boxed{3 + (-9)(x+1) + 4(y-3) + \frac{1}{2}\left[12(x+1)^2 - 18(x+1)(y-3) + 2(y-3)^2\right]}$$

(OK if left unsimplified)

3.d) $f(x,y) = \cos(x-y)$ $\quad (x_0, y_0) = (0,0)$

$f(0,0) = \cos(0) = 1$

$f_x = -\sin(x-y)$ so $f_x(0,0) = 0$

$f_y = +\sin(x-y)$ so $f_y(0,0) = 0$

$f_{xx} = -\cos(x-y)$ so $f_{xx}(0,0) = -1$

$f_{xy} = \cos(x-y)$ so $f_{xy}(0,0) = 1$

$f_{yy} = -\cos(x-y)$ so $f_{yy}(0,0) = -1$

1st degree $\quad 1$

(6) 2nd degree $\quad 1 + \frac{1}{2}\left[(-1)x^2 + 2xy + (-1)y^2\right]$

4 (a) $f(x,y) = 3x^2 - 2xy + y^2$

$f_x = 6x - 2y = 0$ iff $3x = y$

$f_y = -2x + 2y = 0$ iff $x = y$

2 $\Rightarrow$ $(0,0)$ is the only critical point

4 $f_{xx} = 6 \quad f_{xy} = -2 \quad f_{yy} = 2$

so $f_{xx} > 0$, $f_{xx} f_{yy} - (f_{xy})^2 = 12 - 4 = 8 > 0$

$\therefore$ At $(0,0)$, $f(x,y)$ has a <u>local</u> <u>minimum</u>.

(c) $f(x,y) = \frac{x^4}{4} - x^3 + x^2 + 1 - y^2$

<u>crit</u> points:

$f_x = x^3 - 3x^2 + 2x = 0$ so $(x-1)(x-2)x = 0$

4 $f_y = -2y = 0$

Crit points are $(0,0)$, $(1,0)$, $(2,0)$

$f_{xx} = 3x^2 - 6x + 2$

$f_{xy} = 0$

$f_{yy} = -2$

at $(0,0)$ $f_{xx}(0,0) = 2$, $f_{xx} f_{yy} - (f_{xy})^2 = -4 <$

$\Rightarrow$ saddle point for f.

6 at $(1,0)$ $f_{xx}(1,0) = -1 < 0$, $f_{xx} f_{yy} - (f_{xy})^2 = +2$

$\Rightarrow$ local max for f

at $(2,0)$ $f_{xx}(0,0) = 2$ $f_{xx} f_{yy} - (f_{xy})^2 = -4 <$

$\Rightarrow$ saddle point for f.

(e) $f(x,y) = \dfrac{1}{x^2+y^2+1}$

Crit points:

$$f_x = \frac{-2x}{(x^2+y^2+1)^2} = 0$$

$$f_y = \frac{-2y}{(x^2+y^2+1)^2} = 0$$

2 $\Rightarrow \boxed{(x,y) = (0,0)}$

$$f_{xx} = \frac{(x^2+y^2+1)^2(-2) + 2 \cdot 2(x^2+y^2+1)(2x)}{(x^2+y^2+1)^4}$$

$$= \frac{-2x^2-2y^2-2+8x^2}{(x^2+y^2+1)^3} = \frac{6x^2-2y^2-2}{(x^2+y^2+1)^3}$$

$$f_{xy} = \frac{+8xy}{(x^2+y^2+1)^3}$$

$$f_{yy} = \frac{-2x^2+6y^2-2}{(x^2+y^2+1)^3}$$

so $f_{xx}(0,0) = -2$

4 $f_{xy}(0,0) = 0$

$f_{yy}(0,0) = -2$

$f_{xx}(0,0) < 0$ but $f_{xx}f_{yy} - (f_{xy})^2 = 4 > 0$

f has a local <u>max</u> at $(0,0)$.

8. a) $f_x = 2ax + cy + m$ $\quad f_y = cx + 2dy + n$

$f_{xx} = 2a \quad f_{xy} = c \quad f_{yy} = 2d$

so $\ell(x,y) = mx + ny + \ell$

$p_2(x,y) = ax^2 + cxy + dy^2 + mx + ny + \ell \quad (= f(x,y))$

2

(b) at $x = x_0$, $y = y_0$,

2

$$\ell(x,y) = (2ax_0 + cy_0 + m)(x - x_0) + (cx_0 + 2dy_0 + n)(y - y_0)$$

$$+ ax_0^2 + cx_0y_0 + dy_0^2 + mx_0 + ny_0 + \ell$$

$$p_2(x,y) = \ell(x,y) + \frac{1}{2}\Big[2a(x-x_0)^2 + 2c(x-x_0)(y-y_0) + d(y-y_0)^2$$

(c) relation is $\overset{\text{in (b)}}{p_2(x,y)} = f(x,y) = \overset{\text{in (a)}}{p_2(x,y)}$

2 Expanding out: $p_2(x,y)$ at (x_0, y_0) =

reason: $2a\cancel{x_0}x + c\cancel{x}y_0 + mx - 2a\cancel{x_0^2} - c\cancel{x_0y_0} - m\cancel{x_0}$

$+ c\cancel{x_0}y + 2d\cancel{y_0}y + ny - c\cancel{x_0y_0} - 2d\cancel{y_0^2} - n\cancel{y_0}$

$+ a\cancel{x_0^2} + c\cancel{x_0y_0} + d\cancel{y_0^2} + m\cancel{x_0} + n\cancel{y_0} + \ell$

$+ a(x^2 - 2\cancel{xx_0} + \cancel{x_0^2}) + cxy - c\cancel{x_0y} - c\cancel{xy_0} + c\cancel{x_0y_0}$

$+ dy^2 - 2d\cancel{yy_0} + d\cancel{y_0^2}$

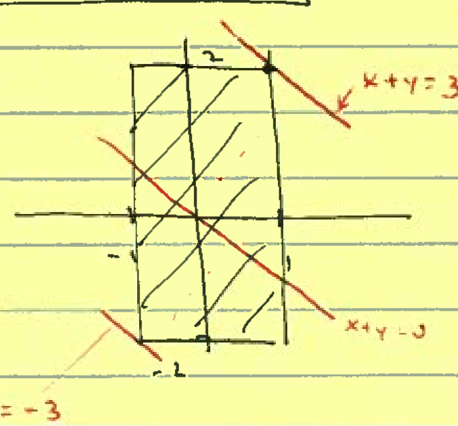
$$= \boxed{ax^2 + cxy + dy^2 + mx + ny + \ell}.$$

§4.3 1. S is the rectangle:

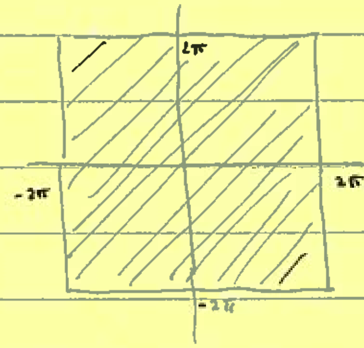
$f(x,y) = x + y$ attains

2 its max $f(1,2) = 3$

min $f(-1,-2) = -3$



(d) $f(x,y) = \sin(x) + \cos(y)$ on

$S =$ 

Because of properties of sin, cos, we have

$-2 \le \sin(x) + \cos(y) \le 2$ for all $(x,y) \in \mathbb{R}^2$.

There are points in the square S where

(4) f takes both of these values:

$$f(\tfrac{\pi}{2}, -2\pi) = f(\tfrac{\pi}{2}, 0) = f(\tfrac{\pi}{2}, 2\pi)$$
$$= f(-\tfrac{3\pi}{2}, -2\pi) = f(-\tfrac{3\pi}{2}, 0) = f(-\tfrac{3\pi}{2}, 2\pi) = 2$$

and $f(\tfrac{3\pi}{2}, \pi) = f(\tfrac{3\pi}{2}, -\pi) = f(-\tfrac{\pi}{2}, \pi) = f(-\tfrac{\pi}{2}, -\pi) = -2.$

only (-1) if they do not find all of them

2 (a) the Lagrange equations

$$2\begin{cases} 2x = 2\lambda x \\ -4y = 2\lambda y \\ x^2+y^2-1 = 0 \end{cases}$$

If we multiply the first equation by y and the second by x, then

$$2xy = 2\lambda xy = -4xy$$

so $xy = 0$, and $x=0$ or $y=0$

2 $\Rightarrow (x,y) = (\pm 1, 0), \ (0, \pm 1)$.

2 $f(\pm 1, 0) = 1, \ f(0, \pm 1) = -2$, so

$f(x,y)$ has ~~maximum~~ value 1 on the constraint curve at $(\pm 1, 0)$, and minimum value -2 at $(0, \pm 1)$.

(d) $f(x,y) = (x-1)^2 + 4y^2$

$g(x,y) = 2x^2 + y^2 - 3$

Lagrange Equations:

$$\{2(x-1) = 4\lambda x\} \cdot y$$

2 $$\{8y = 2\lambda y\} \cdot 2x$$

$$2x^2 + y^2 - 3 = 0$$

Eliminating λ between the 1st two equations:

$$2(x-1)y = 4\lambda xy = 16xy$$

So $$2xy - 2y = 16xy$$

$$14xy + 2y = 0$$

$$(7x+1)y = 0$$

2 so $x = -\frac{1}{7}$ or $y = 0$

$$3 - \frac{2}{49} = \frac{147-2}{49} = \frac{145}{49}$$

If $x = -\frac{1}{7}$, $\frac{2}{49} + y^2 - 3 = 0$

$$y = \pm\sqrt{\frac{145}{49}} = \frac{\pm\sqrt{145}}{7}$$

If $y = 0$, $x = \pm\sqrt{\frac{3}{2}}$

$\begin{array}{r}145\\ \times\ 4\\ \hline 580\end{array}$

At $\left(-\frac{1}{7}, \pm\frac{\sqrt{145}}{7}\right)$, $f\left(-\frac{1}{7}, \pm\frac{\sqrt{145}}{7}\right) = \frac{64}{49} + \frac{580}{49} = \frac{644}{49}$ ← max

At $\left(\sqrt{\frac{3}{2}}, 0\right)$ $f\left(\sqrt{\frac{3}{2}}, 0\right) = \left(\sqrt{\frac{3}{2}} - 1\right)^2 = \frac{(\sqrt{3}-2)^2}{4}$

$= \frac{7-4\sqrt{3}}{4}$ ← min

4 $\left(-\sqrt{\frac{3}{2}}, 0\right)$ $f\left(-\sqrt{\frac{3}{2}}, 0\right) = \left(-\frac{\sqrt{3}}{2} - 1\right)^2 = \frac{(\sqrt{3}+2)^2}{4} = \frac{7+4\sqrt{3}}{4}$

3 a) $f(x,y) = x^2 + y^2 - 2x + 1 = (x-1)^2 + y^2$ $S = \{(x,y) \mid x^2 + y^2 \leq 4\}$

Crit points of f:

4

$\frac{\partial f}{\partial x} = 2x - 2 = 0 \Rightarrow x = 1$

$\frac{\partial f}{\partial y} = 2y = 0 \Rightarrow y = 0$

$\frac{\partial^2 f}{\partial x^2} = 2$ $\frac{\partial^2 f}{\partial x \partial y} = 0$ $\frac{\partial^2 f}{\partial y^2} = 2$ so $(1,0)$ is a local minimum

$f(1,0) = 0$

on boundary:

$$\begin{cases} 2x - 2 = \lambda \cdot 2x \\ 2y = 2\lambda y \\ x^2 + y^2 = 4 \end{cases}$$

6

Second equation: $y = 0$ or $\lambda = 1$

If $y = 0$, $x = \pm 2$: $(2,0)$, $(-2,0)$

If $\lambda = 1$, 1st equation says $-2 = 0$ contradiction.

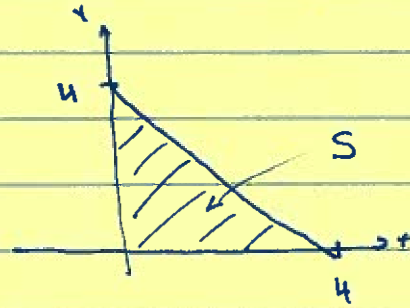
$f(2,0) = 1$

$f(-2,0) = 9$

so $\boxed{\begin{array}{l}\text{min at } (1,0) \\ \text{max at } (-2,0)\end{array}}$ $\boxed{\begin{array}{l}f(1,0) = 0 \\ f(-2,0) = 9\end{array}}$

(d) $f(x,y) = x^2y^2 - 2x - 4y + 5$

$S = \{ (x,y) \mid x \geq 0,\ y \geq 0,\ x+y \leq 4 \}$



Crit points of f:

4

$f_x = 2xy^2 - 2 = 0$ so $xy^2 = 1$

$f_y = 2yx^2 - 4 = 0$ so $x^2y = 2$

$\Rightarrow x = 2y$

so $y = \sqrt[3]{1/2} = 1/\sqrt[3]{2}$

$x = 2\sqrt[3]{1/2} = \sqrt[3]{4}$

$(1/\sqrt[3]{2}, \sqrt[3]{4}) \doteq (.79, 1.59)$

on $x = 0$:

$2xy^2 - 2 = \lambda \cdot 1$

$2yx^2 - 4 = 0$

$x = 0$

inconsistent (no sols)

on $y = 0$:

$2xy^2 - 2 = 0$

$2yx^2 - 4 = \lambda$

$y = 0$

inconsistent (no sols)

on $\underline{x+y=4}$:

$$\left.\begin{aligned} 2xy^2 - 2 &= \lambda \\ 2yx^2 - 4 &= \lambda \end{aligned}\right] \qquad xy^2 - 1 = yx^2 - 2$$

$$x + y = 4$$

so $\begin{cases} x^2y - xy^2 = 1 \\ x + y = 4 \end{cases} \qquad y = 4 - x$

$\Rightarrow \quad x^2(4-x) - x(4-x)^2 = 1$

$$4x^2 - x^3 - 16x + 8x^2 - x^3 = 1$$

so $\quad -2x^3 + 12x^2 - 16x - 1 = 0$

(from maple): roots are approx:

$$x = -.06, \quad 2.13, \quad 3.93$$

Ignore 1st since $x \geq 0$.

6

so $\quad (x,y) \doteq (2.13, 1.87), \ (3.93, .07)$

$$\begin{cases} f(.79, 1.59) \doteq -1.35 \\ f(2.13, 1.87) \doteq 9.13 \leftarrow \text{max} \end{cases}$$

$f(3.93, .07) \doteq -3.06 \leftarrow$ min

5. (a) ∇f is orthogonal to the constraint curve at (approx.)

4 $(-1.5, 1)$, $(-.2, -1)$, $(1, -1.4)$, $(-.2, 1)$, $(.8, .4)$, $(1.2, -1.4)$

(b) at $(-1, 0)$, $(-1.5, 1.5)$, $(.5, 0)$, $(-1.5, -1.5)$

only -1 if they don't find all of them and anything within $\pm .5$ is ok for estim

9 9. (a) S is not closed since $(0,0)$ is

2 a boundary point of S, but $(0,0) \notin S$

(b) If $\varepsilon > 0$, then $f(x,y) = 1 - x^2 - y^2 = 1 - \varepsilon$ for all (x,y) satisfying $x^2 + y^2 = \varepsilon$ so $x^2 + y^2 = (\sqrt{\varepsilon})^2$.

2 this circle of radius $\sqrt{\varepsilon}$ is contained in S whenever $0 < \varepsilon \le 1$. However, the only point where $f(x,y) = 1$ is $(x,y) = (0,0)$ and $(0,0) \notin S$.

10. (a) $S = \{(x,y) \mid x^2 + y^2 \le 1\}$ is closed, since the boundary of S $(= \{(x,y) \mid x^2 + y^2 = 1\})$ is contained in S.

6 (b) $\lim_{(x,y) \to (0,0)} f(x,y) = 1$, but $f(0,0) = 0$.

Hence $f(x,y)$ is not continuous at $(0,0)$.

(c) is same idea as 9b ← ok if they just say that

4.4 1(c) $f(x,y,z) = xyz$

$g(x,y,z) = (x-1)^2 + y^2 + z^2 - 1$

Lagrange Equations:

$$\begin{cases} yz = 2\lambda(x-1) & (1) \\ xz = 2\lambda y & (2) \\ xy = 2\lambda z & (3) \\ (x-1)^2 + y^2 + z^2 - 1 = 0 & (4) \end{cases}$$

multiply (2) by z, (3) by y equate:

$$xz^2 = xy^2 \Rightarrow \boxed{x=0} \text{ or } \boxed{z = \pm y}$$

multiply (1) by y, (2) by $(x-1)$ equate:

$$y^2 z = x(x-1)z \Rightarrow \boxed{z=0} \text{ or } \boxed{y^2 = x(x-1)}$$

Cases:

I. $x=0$ and $z=0 \Rightarrow y=0$ from constraint $(0,0,0)$

II. $x=0$ and $y^2 = x(x-1) \Rightarrow y=0$ so $z=0$ $(0,0,0)$

III. $z = \pm y$ and $z=0 \Rightarrow y = z = 0$ so $x = 0, 2$ $\begin{cases}(0,0,0)\\(2,0,0)\end{cases}$

IV. $z = \pm y$ and $y^2 = x(x-1)$

in constraint:

$$(x-1)^2 + x(x-1) + x(x-1) = 1$$

$$3x^2 - 4x + 1 = 1$$

$3x^2 - 4x = 0$

So $x = 0$ or $x = \frac{4}{3}$

- $x = 0 \Rightarrow y = 0 \Rightarrow z = 0$
- $x = \frac{4}{3} \Rightarrow y^2 = \frac{4}{3} \cdot \frac{1}{3} = \frac{4}{9}$, so $y = \pm \frac{2}{3}$

$\frac{1}{9} + \frac{4}{9} + z^2 = 1 \Rightarrow z = \pm \frac{2}{3}$

this gives $\left(\frac{4}{3}, \frac{2}{3}, \frac{2}{3}\right)$, $\left(\frac{4}{3}, -\frac{2}{3}, \frac{2}{3}\right)$, $\left(\frac{4}{3}, \frac{2}{3}, -\frac{2}{3}\right)$,

$\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$.

$f(0,0,0) = f(2,0,0) = 0$

$f\left(\frac{4}{3}, \frac{2}{3}, \frac{2}{3}\right) = f\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right) = \frac{16}{27}$ ← max

12 $f\left(\frac{4}{3}, \frac{2}{3}, -\frac{2}{3}\right) = f\left(\frac{4}{3}, -\frac{2}{3}, \frac{2}{3}\right) = -\frac{16}{27}$ ← min.

2 (b) $f(x,y,z) = x^2 + yz$

$S = \{ x^2 + y^2 + z^2 \le 3 \}$

Crit points: $f_x = 2x = 0 \Rightarrow (0,0,0)$

$f_y = z = 0$

$f_z = y = 0$

Lagrange:

$2x = 2\lambda x \qquad x^2 + y^2 + z^2 = 3$

$z = 2\lambda y$

$y = 2\lambda z$

1^{st} says $x=0$ or $\lambda=1$

$\lambda=1 \Rightarrow z=2y$ and $y=2z$ so $y=z=0$

and $x=\pm\sqrt{3}$ $(\pm\sqrt{3},0,0)$

$x=0$:

$$\left.\begin{array}{l} z=2\lambda y \\ y=2\lambda z \end{array}\right] \Rightarrow z^2=y^2$$

$y^2+z^2=3$

gives $\left(0,\sqrt{\tfrac{3}{2}},\sqrt{\tfrac{3}{2}}\right)$, $\left(0,-\sqrt{\tfrac{3}{2}},-\sqrt{\tfrac{3}{2}}\right)$

$\left(0,\sqrt{\tfrac{3}{2}},-\sqrt{\tfrac{3}{2}}\right)$, $\left(0,-\sqrt{\tfrac{2}{2}},\sqrt{\tfrac{3}{2}}\right)$

$f(0,0,0)=0$

$f(\pm\sqrt{3},0,0)=3 \leftarrow \boxed{max}$

12 $f\left(0,\sqrt{\tfrac{3}{2}},\sqrt{\tfrac{3}{2}}\right)=f\left(0,-\sqrt{\tfrac{3}{2}},-\sqrt{\tfrac{3}{2}}\right)=\tfrac{3}{2}$

$f\left(0,-\sqrt{\tfrac{3}{2}},\sqrt{\tfrac{3}{2}}\right)=f\left(0,+\sqrt{\tfrac{3}{2}},-\sqrt{\tfrac{3}{2}}\right)=-\tfrac{3}{2} \leftarrow \boxed{min}$