Mathematics 241, section 1 – Multivariable Calculus
Solutions for Exam III – December 3, 2010

I.

A) (10) Give a precise statement of the Second Derivative Test for local extrema of functions of two variables.

Solution: Let (x_0, y_0) be a critical point of $f(x, y)$ and assume the partial derivatives f_x and f_y are continuous in some open set containing (x_0, y_0). (Note: this implies that $f(x, y)$ is differentiable at (x_0, y_0).) If

1) $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
2) $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
3) $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 < 0$, then f has a saddle point at (x_0, y_0) (neither a maximum nor a minimum).
4) If $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 = 0$, then the test gives no conclusion.

B) (20) Let $f(x, y) = 3x^4 + y^3 - 4x^3 - 3y$. Find all critical points of f and apply the Second Derivative Test to each.

1. *Solution:* The partial derivatives are

$$\begin{aligned} f_x &= 12x^3 - 12x^2 = 12x^2(x-1) \\ f_y &= 3y^2 - 3 \end{aligned}$$

So the critical points are the solutions of $f_x = f_y = 0$, or

$$(0, 1), (0, -1), (1, 1), (1, -1).$$

Next, we compute:

$$\begin{aligned} f_{xx} &= 36x^2 - 24x \\ f_{xy} &= 0 \\ f_{yy} &= 6y. \end{aligned}$$

At $(0, \pm 1)$,

$$f_{xx}(0, \pm 1)f_{yy}(0, \pm 1) - (f_{xy}(0, \pm 1))^2 = 0 \cdot (\pm 6) - 0^2 = 0.$$

Hence the Second Derivative Test gives *no conclusion* at those points. (By looking at the form of the function, we can see, in fact, that both of these are saddle points of a sort – they are neither maxima nor minima.) At $(1,1)$, $f_{xx}(1,1)f_{yy}(1,1)-(f_{xy}(1,1))^2 = 12\cdot 6 - 0 = 72 > 0$ and $f_{xx}(1,1) = 12 > 0$. Therefore, f has a *local minimum* at $(1,1)$. Finally, at $(-1,1)$, $f_{xx}(1,-1)f_{yy}(1,-1)-(f_{xy}(1,-1))^2 = 12\cdot -6 - 0 = -72 < 0$. Therefore f has a *saddle point* at $(1,-1)$.

II.

A) (20) Find the maximum and minimum values of

$$f(x,y) = xy - y$$

on the closed and bounded set

$$S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 4\}.$$

Solution: We have

$$\begin{aligned} f_x &= y \\ f_y &= x - 1 \end{aligned}$$

So f has a critical point at $(1,0) \in S$. By the Second Derivative Test, we can see that f has a saddle point here, and $f(1,0) = 0$. There must be smaller and larger values on the boundary, so we use Lagrange Multipliers next. With the constraint equation $x^2 + y^2 = 4$, the Lagrange equations are

$$\begin{aligned} y &= 2\lambda x \\ x - 1 &= 2\lambda y \\ x^2 + y^2 - 4 &= 0. \end{aligned}$$

If we multply the first equation by y and the second by x, then we can equate the right sides to obtain an equation in which λ has been eliminated: $y^2 = x(x-1) = x^2 - x$. We can then use this equation to substitute for the y^2 and eliminate y in the constraint equation:

$$2x^2 - x - 4 = 0.$$

This equation can be solved by the quadratic formula:

$$x = \frac{1 \pm \sqrt{1+32}}{4} \doteq 1.69, -1.19.$$

Then from the constraint equation we get two y-values for each of these x-values, so there are four solutions of the Lagrange equations all together – approximately

$$(-1.19, \pm 1.61), \text{ and } (1.69, \pm 1.08).$$

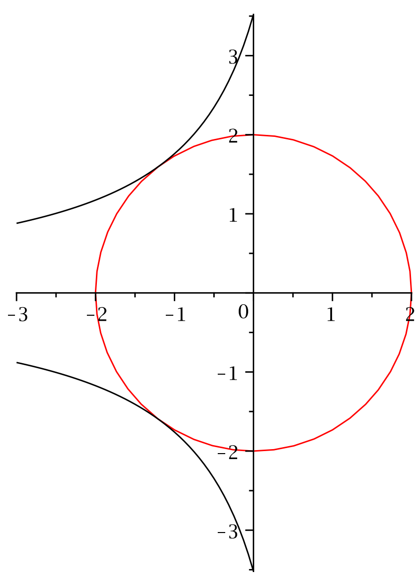
We have

$$\begin{aligned}
f(-1.19,-1.61) &\doteq 3.52 \\
f(-1.19,1.61) &\doteq -3.52 \\
f(1.69,-1.08) &\doteq -.74 \\
f(1.69,1.08) &\doteq .74
\end{aligned}$$

So putting everything together, we see that the maximum value of f on S is about 3.52 and the minimum value of f on S is about -3.52.

B) (5) Give a sketch of the contours of f for the maximum and minimum values you found in part A.

Solution: The function is $f(x,y) = (x-1)y$. Hence the contours of f for values not equal to zero are *rectangular hyperbolas* with asymptotes equal to the lines $x = 1$ and $y = 0$.



The sketch should show (branches of) two hyperbolas *tangent to* the boundary circle of S passing through the points where the maximum and minimum values are attained. The relevant portions of the branches of the hyperbolas are in the second and third quadrants of the plane. (Recall the geometry of Lagrange Multipliers we saw in Lab 4.)

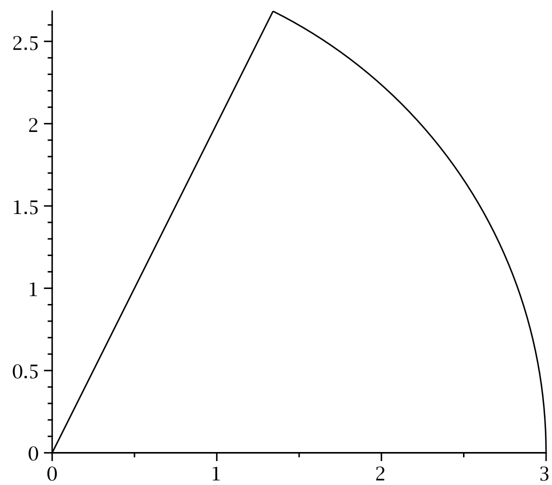
III. All parts of this problem refer to the region

$$R = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9, y \geq 0, y \leq 2x\}.$$

and the solid $S \subset \mathbb{R}^3$ with "shadow" R in the xy-plane and $0 \leq z \leq x^2$.

A) (5) Sketch the region R in the plane.

Solution: This is the region inside a circle of radius 3, above the x-axis, and below the line $y = 2x$:



B) (10) Set up iterated integral(s) of the form $\int\int f(x, y)\ dy\ dx$ to compute the volume of S. Do not compute the value.

Solution: Since the top boundary changes when $y = 2x$ crosses the circle, we need the sum of two interated integrals. The line $y = 2x$ crosses $x^2 + y^2 = 9$ when $5x^2 = 9$ so $x = 3/\sqrt{5}$.

$$\int_0^{3/\sqrt{5}} \int_0^{2x} x^2\ dy\ dx + \int_{3/\sqrt{5}}^{3} \int_0^{\sqrt{9-x^2}} x^2\ dy\ dx.$$

C) (10) Now set up iterated integral(s) of the form $\int\int f(x, y)\ dx\ dy$ to compute the volume of S. Do not compute the value.

Solution: This one is slightly tricky because you have to notice that, even though the integral does not need to be split, the region R does not extend up to $y = 3$. The range of y-values is determined like this: The line $y = -2x$ crosses $x^2 + y^2 = 9$ when $5x^2 = 9$ so $x = 3/\sqrt{5}$ and $y = 6/\sqrt{5}$. The integral is:

$$\int_0^{6/\sqrt{5}} \int_{y/2}^{\sqrt{9-y^2}} x^2\ dx\ dy.$$

D) (10) Finally, set up a polar coordinate integral that will compute the volume of S. Do not compute the value.

Solution: With polar coordinates, you do not need to split into two integrals, and the limits of integration are all constant. The line $y = 2x$ is the ray $\theta = \arctan(2)$. With the "extra r" from the dA in polar coordinates:

$$\int_0^{\arctan(2)} \int_0^3 r^3 \cos^2\theta \; dr \; d\theta.$$

E) (10) Finally, compute the volume of S using any one of your answers from parts B,C,D (your choice).

Solution: Don't know about you, but the first two look pretty bad to me! Let's evaluate the last one:

$$\begin{aligned}
\int_0^{\arctan(2)} \int_0^3 r^3 \cos^2\theta \; dr \; d\theta &= \int_0^3 r^3 \; dr \cdot \int_0^{\arctan(2)} \cos^2\theta \; d\theta \\
&= \left.\frac{r^4}{4}\right|_0^3 \cdot \left.\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right)\right|_0^{\arctan(2)} \\
&= \frac{81}{4} \cdot \left(\frac{1}{2}\arctan(2) + \frac{1}{5}\right).
\end{aligned}$$

Extra Credit (10) Evaluate the improper double integral

$$\int\int_{\mathbb{R}^2} e^{-x^2-y^2} \; dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} \; dy \; dx.$$

We can evaluate this by switching to polar coordinates! The integral over all of $\mathbb{R}^2$ is

$$\int_0^{2\pi} \int_0^{\infty} e^{-r^2} r \; dr \; d\theta$$

This equals

$$2\pi \cdot \lim_{B\to\infty} \left. -\frac{1}{2}e^{-r^2}\right|_0^B = 2\pi \cdot \lim_{B\to\infty} \frac{1}{2} - \frac{1}{2}e^{-B^2} = \pi.$$

Wow!