Mathematics 241 – Multivariable Calculus
Midterm 3 – Solutions
November 30

I.

A) (15) Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = (xz - y^2, xy + z, 2x + \sin(y)).$$

Compute the divergence $\mathrm{div}(\mathbf{F})$ and the curl $\mathrm{curl}(\mathbf{F})$.

Solution: Using the ∇ operator, the divergence is

$$\mathrm{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xz - y^2) + \frac{\partial}{\partial y}(xy + z) + \frac{\partial}{\partial z}(2x + \sin(y)) = z + x.$$

The curl is

$$\mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = (\cos(y) - 1, x - 2, 3y).$$

B) (5) Show that if $\mathbf{F} = (F_1, F_2, F_3)$ is any vector field on a domain in $\mathbf{R}^3$ whose component functions have continuous second-order partial derivatives, then $\mathrm{div}(\mathrm{curl}(\mathbf{F})) = 0$.

Solution:

$$\begin{aligned}
\mathrm{div}(\mathrm{curl}(\mathbf{F})) &= \mathrm{div}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, -\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right), \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(-\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\
&= \frac{\partial^2 F_3}{\partial y \partial x} - \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_3}{\partial x \partial y} + \frac{\partial^2 F_1}{\partial z \partial y} + \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_1}{\partial y \partial z} \\
&= 0
\end{aligned}$$

by the equality of the mixed partial derivatives.

II. All parts of this problem refer to the function

$$f(x, y) = x^3 + y^2 - 6xy + 6x + 3y.$$

A) (10) Determine all critical points of f.

Solution: The critical points are the solutions of

$$\begin{aligned}
0 = f_x &= 3x^2 - 6y + 6 \\
0 = f_y &= 2y - 6x + 3
\end{aligned}$$

Solving for y in the second equation, $y = 3x - 3/2$. Substituting into the first equation, $0 = 3x^2 - 18x + 15 = 3(x - 1)(x - 5)$ and $x = 1$ or $x = 5$. The critical points are

$$(1, 3/2), \quad (5, 27/2).$$

B) (15) Use the Hessian Criterion (second derivative test) to classify each of the critical points as a local maximum, a local minimum, a saddle point, or a degenerate critical point.

Solution: The second derivatives of f are

$$f_{xx} = 6x, \quad f_{xy} = -6, \quad f_{yy} = 2.$$

- At $(1, 3/2)$, the Hessian matrix is

$$D^2(f)(1,3/2) = \begin{pmatrix} 6 & -6 \\ -6 & 2 \end{pmatrix}$$

  Since $\det(D^2(f)(1,3/2)) = 12 - 36 = -24 < 0$, this is a *saddle point.*

- At $(5, 27/2)$, the Hessian matrix is

$$D^2(f)(5,27/2) = \begin{pmatrix} 30 & -6 \\ -6 & 2 \end{pmatrix}$$

  Since $\det(D^2(f)(5,7/2)) = 60 - 36 = 24 > 0$ and $30 > 0$, this is a *local minimum.*

III. (20) A mountain has the shape of the graph $z = f(x,y) = 4 - x^2 - \frac{y^2}{4}$, where x, y are east-west and north-south map coordinates respectively, and z is the altitude (in thousands of feet). If you hike along a trail on the mountain described by the circle $g(x,y) = x^2 + y^2 = 1$ on the map, what are the highest and lowest altitudes that you reach?

Solution: We want to maximize and minimize $f(x,y)$ on the constraint curve $g(x,y) = 1$, so this is a job for Lagrange multipliers:

$$\begin{aligned} f_x = -2x &= 2\lambda x = \lambda g_x \\ f_y = -y/2 &= 2\lambda y = \lambda g_y \\ x^2 + y^2 &= 1 \end{aligned}$$

From the x equation, we see that either $x = 0$ or $\lambda = -1$. If $x = 0$, then $y = \pm 1$ from the constraint equation. If $\lambda = -1$, then the y equation gives $y = 0$, and $x = \pm 1$ from the constraint. Hence there are four points that solve the LM equations: $(1,0), (-1,0), (0,1), (0,-1)$. The altitudes at these points are $f(1,0) = f(-1,0) = 3$ (thousand feet) and $f(0,1) = f(0,-1) = 3.75$ (thousand feet). The maximum altitude is 3750 and the minimum altitude is 3000 feet at the points on the trail.

IV. All parts of this problem refer to the region D in the right half-plane ($x \geq 0$), bounded by $y = x$, $y = -x$ and $x^2 + y^2 = 9$.

A) (5) Sketch the region D.

Solution: The region is a *quarter circle*, with the straight edges along the two given lines and curved edge along the circle of radius 3, center $(0,0)$.

B) (10) What would the limits of integration be to compute $\int\int_D f(x,y)\ dx\ dy$ integrating with respect to y first, then x?

Solution: This way, we must split the region along $x = 3/\sqrt{2}$.

$$\int_0^{3/\sqrt{2}} \int_{-x}^{x} f(x,y)\ dydx + \int_{3/\sqrt{2}}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x,y)\ dydx.$$

C) (10) What would the limits of integration be to compute $\int\int_D f(x,y)\ dx\ dy$ integrating with respect to x first, then y?

Solution: We must split the region along the line $y = 0$

$$\int_{-3/\sqrt{2}}^{0} \int_{-y}^{\sqrt{9-y^2}} f(x,y)\ dxdy + \int_0^{3/\sqrt{2}} \int_y^{\sqrt{9-y^2}} f(x,y)\ dxdy.$$

D) (10) Using any applicable method, compute $\int\int_D x\ dx\ dy$.

Solution 1: Changing to polar coordinates this becomes

$$\begin{aligned}
\int_{-\pi/4}^{\pi/4} \int_0^3 (r\cos\theta) r\ drd\theta &= \int_0^3 r^2\ dr \cdot \int_{-\pi/4}^{\pi/4} \cos\theta\ d\theta \\
&= \left.\frac{r^3}{3}\right|_0^3 \cdot \sin\theta\Big|_{-\pi/4}^{\pi/4} \\
&= 9\sqrt{2}.
\end{aligned}$$

Solution 2: If we use the form from B above, the first integral is

$$\begin{aligned}
\int_0^{3/\sqrt{2}} \int_{-x}^{x} x\ dydx &= \int_0^{3/\sqrt{2}} xy\Big|_{y=-x}^{y=x}\ dx \\
&= \int_0^{3/\sqrt{2}} 2x^2\ dx \\
&= \left.\frac{2x^3}{3}\right|_0^{3/\sqrt{2}} \\
&= \frac{9\sqrt{2}}{2}.
\end{aligned}$$

The second integral is evaluated using a u-subsitution after the first integration (the integral from the table is *not* needed, or even applicable):

$$\begin{aligned}\int_{3/\sqrt{2}}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x\ dydx &= \int_{3/\sqrt{2}}^{3} 2x\sqrt{9-x^2}\ dx \\ &= -\frac{2}{3}(9-x^2)^{3/2}\Big|_{3/sqrt2}^{3} \\ &= \frac{2}{3}(9/2)^{3/2} \\ &= \frac{2}{3}\frac{3^3}{2\sqrt{2}} = \frac{9\sqrt{2}}{2}.\end{aligned}$$

Then the integral is the sum

$$\frac{9\sqrt{2}}{2}+\frac{9\sqrt{2}}{2} = 9\sqrt{2}.$$

Extra Credit (10) Compute the triple integral of the function $f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$ over the unit ball $x^2+y^2+z^2 \leq 1$ in $\mathbf{R}^3$. (A good choice of coordinates is essential here!)

Solution: Spherical coordinates are the best choice. The integral to be computed is

$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 e^{\rho^3}\rho^2 \sin\phi\ d\rho\ d\phi\ d\theta.$$

Since the function is a product of functions of ρ, ϕ, and θ we can use the "product trick":

$$= 2\pi\int_0^{\pi}\sin\phi\ d\phi\int_0^1 \rho^2 e^{\rho^3}\ d\rho = 4\pi\int_0^1 \rho^2 e^{\rho^3}\ d\rho.$$

For the last integral here, let $u=\rho^3$, then $du = 3\rho^2 d\rho$ and the integral is just $\int e^u du$ up to a constant. The final answer is

$$\frac{4\pi}{3}(e-1).$$