Mathematics 241 – Multivariable Calculus
Lab Day 4 – Constrained Optimization and the Geometry of Lagrange Multipliers
November 5, 2007

Background

In last Friday's class, we started to investigate the problem of determining where a function $f(x, y)$ took its maximum and minimum values on curve in the plane $g(x, y) = 0$ (such as the boundary of a compact region). The general name for this sort of problem is *constrained optimization*, and the curve $g(x, y) = 0$ is often called the *constraint set* in this context. Today, we want to look at some examples of this, and try to see the geometry behind a method for solving problems of this form – the *Method of Lagrange multipliers*.

Lab Work, Part 1

In the first parts of this lab you will be working through a collection of examples designed to help you visualize what is happening when we look at a maxima and minima of a function $f(x, y)$ on a constraint set $g(x, y) = 0$. The goal is to try to see a connection between the constraint set and the *level curves* of the function $f(x, y)$ when $f(x, y)$ reaches a maximum or minimum on the constraint set.

Example 1

Enter the following commands as directed and answer questions A,B,C in text regions in your Maple worksheet. Our first example will study the function

$$f(x, y) = x^2 + \frac{y^2}{2}$$

on the constraint set defined by

$$g(x, y) = \frac{x^2}{2} + y^2 - 1 = 0$$

(an ellipse). First we will construct a composite Maple plot of the level curves of f together with the constraint curve. To plot the level curves of f superimposed on the constraint curve, enter the commands

```
with(plots):
fPlot1:=contourplot(x^2 + y^2/2, x=-3..3,y=-3..3,contours=31):
gPlot1:=plot([sqrt(2)*cos(t),sin(t),t=0..2*Pi],color=blue):
display(fPlot1,gPlot1,scaling=constrained);
```

What is this doing?

- The `contourplot` command generates a plot of the level curves and saves the specification of the plot under the name `fPlot`. It plots the portions of 31 level curves contained in the square in the plane with $-3 \le x, y \le 3$. Maple chooses the c-values for the level curves here, and those are *not displayed in the graph.* In drawing this graph, Maple selects c-values to use to display as many level curves as possible "filling out" the square. It is also possible to specify the c-values we want. We'll see how to do that later.

- The `plot` command draws the constraint curve, using a standard set of parametric equations for the ellipse, and saves that under the name `gPlot`.
- The `display` command superimposes the two plots on the same set of coordinate axes.

If you look at the form of the function $f(x, y)$, you will see that the level curves of f are ellipses also, and the farther the ellipse is out from the origin, the larger the c-value would be in the equation $f(x, y) = c$ defining the level curve.

A) By examining your plot, try to determine where $f(x, y)$ reaches its minimum and maximum values on the constraint curve. Give approximate (x, y)-coordinates of those points by "eyeballing" the composite plot of the level curves and the constraint curve.

Next, we want to look at the function $f(x, y)$ *restricted to the constraint curve* in a different way. As we saw above in plotting the constraint curve, it is natural to describe that ellipse by the parametric equations:

$$x = \sqrt{2}\cos(t), \quad y = \sin(t).$$

Let's see what happens if we substitute these into $f(x, y)$ to obtain

$$f\left(\sqrt{2}\cos(t), \sin(t)\right) = \left(\sqrt{2}\cos(t)\right)^2 + \frac{(\sin(t))^2}{2} = 2\cos^2(t) + \frac{\sin^2(t)}{2}.$$

Plot this function of t on the interval $0 \le t \le 2\pi$ using the following Maple command:

```
plot(2*cos(t)^2 + sin(t)^2/2, t = 0..2*Pi);
```

The interval $0 \le t \le 2\pi$ of parameter values takes us once around the constraint ellipse in the counterclockwise direction, starting from $(\sqrt{2}, 0)$ when $t = 0$.

B) How does this new plot correspond to what you said in part A? Give approximate t values corresponding to all local maxima and local minima of f on the constraint curve and indicate which (x, y) points from part A correspond to which.

Now for the most important point. Look at the graph of f restricted to the constraint curve. You should see that the maximum value f-value attained is apparently 2, and the minimum is apparently $1/2$. Let's actually go back and draw *just the level curves for those two c-values* together with the constraint curve and try to understand what is happening. To do this, enter the commands

```
newfPlot1:=contourplot(x^2 + y^2/2, x=-3..3,y=-3..3,contours=[0.5,2]):
        display(newfPlot1,gPlot1,scaling=constrained);
```

This will display the level curves for just the two given c-values, $c = 0.5$ and $c = 2$, superimposed on the plot of the constraint curve. Note, the command defining `gPlot1` from before saved the result under that name so we can reuse it here without having to generate that plot again.

C) What is true about these two contours and the constraint curve where they meet? How would you describe the way they meet?

Example 2

Modify the commands given above to generate plots for the function

$$f(x, y) = x^2 - 2y^2$$

on the constraint set

$$g(x, y) = (x - 1)^2 + (y - 1)^2 - 1 = 0.$$

First generate a plot showing 31 level curves of f superimposed on the constraint set. Note that the constraint set is a circle of radius 1 centered at the point $(1, 1)$. The most convenient parametric equations to use for the constraint curve will be

$$x = 1 + \cos(t) \qquad y = 1 + \sin(t).$$

Now the interval $0 \leq t \leq 2\pi$ takes you once around the circle in the counterclockwise direction starting from the point $(2, 1)$. Then, generate a plot of $f(1 + \cos(t), 1 + \sin(t))$ and identify the t-values where the local maxima and minima occur and the corresponding (x, y). Finally, determine the c-values for the level curves of f at the maximum and minimum points, and plot those level curves together with the constraint set. Answer questions A,B,C in Example 1 above for this example in text regions in this section of your Maple worksheet.

Example 3

Here is a somewhat more elaborate example. Consider the function

$$f(x, y) = y^2 - x^3 + x^2$$

on the constraint curve with equation:

$$g(x, y) = 250x^2 - 930xy + 1521y^2 - 729 = 0.$$

As you might recognize, the constraint curve is a rotated ellipse. It can be given in parametric form as:

$$x = \cos(t) + \frac{12}{5}\sin(t) \qquad y = -\frac{1}{3}\cos(t) + \sin(t).$$

Here we want to start in directly by studying the restriction of the function $f(x, y)$ to the constraint curve. Let's plot that to begin with. Enter the following commands to define the function $f(x, y)$, then substitute $x = \cos(t) + \frac{12}{5}\sin(t)$ and $y = -\frac{1}{3}\cos(t) + \sin(t)$ into $f(x, y)$, then plot the substituted function:

```
f := y^2 - x^3 + x;
foncon := subs({x = cos(t) + 12*sin(t)/5,y=-cos(t)/3 + sin(t)},f);
plot(foncon,t=0..2*Pi);
```

You should see a very "interesting" graph with *six* critical points – three local maxima and three local minima.

A) Estimate the t-values where these occur and the values of f at the critical points from the graph.

Now we want to understand how the level curves of f are meeting the constraint curve at the locations of these local maxima and minima.

B) As in part C of Examples 1 and 2 above, plot the level curves of $f(x, y)$ for the c-values coming from the values of $f(x, y)$ at the local maxima and minima from part A here. Combine with a parametric plot of the constraint curve to produce a picture like the ones from part C of Examples 1 and 2. What is true about the way the level curve of $f(x, y)$ meets the constraint curve in each case?

Lab Work, part 2 – Lagrange multipliers

The geometry we are seeing in Examples 1, 2, and 3 is the basis for the *method of Lagrange multipliers* for solving constrained optimization problems. As we will see in class on Tuesday November 6, the idea is that the geometry of the way the level curves of the objective function $f(x, y)$ met the constraint curve at the maximum or minimum points in those examples is a general phenomenon. It can also be generalized to any number of variables. This leads to the the following condition. In the plane, if $f(x, y)$ has a local maximum or minimum on the constraint curve at $(x, y) = (a, b)$, then

$$\nabla f(a, b) = \lambda \nabla g(a, b)$$

for some scalar λ called the *Lagrange multiplier.* (The same equation holds at any critical point of the restriction of $f(x, y)$ to the constraint curve.) Similarly, if $f(x, y, z)$ has a local maximum or minimum on the constraint surface $g(x, y, z) = 0$ at $(x, y, z) = (a, b, c)$, then

$$\nabla f(a, b, c) = \lambda \nabla g(a, b, c).$$

To find these points we can set up the system of equations

$$\begin{aligned} f_x(x, y, z) &= \lambda g_x(x, y, z) \\ f_y(x, y, z) &= \lambda g_y(x, y, z) \\ f_z(x, y, z) &= \lambda g_z(x, y, z) \\ g(x, y, z) &= 0 \end{aligned} \tag{1}$$

and try to solve for (x, y, z, λ). For relatively simple problems like some of the questions on this week's problem set from section 4.3, this can be done "by hand." More complicated examples are often solved using computers and numerical methods for solving systems of equations like the features of Maple we will see next.

Worked Example 4

The problem is to find the critical points of $f(x, y, z) = 3xy - 4z^2 + x$ on the constraint surface defined by $3x + y - 2xz - 1 = 0$. First we define the functions f and g using:

```
f := 3*x*y-4*z^2+x;
g := 3*x+y-2*x*z-1;
```

Then the Lagrange multiplier equations (1) can be set up using a command like this:

```
LMEqs := {diff(f,x)-lambda*(diff(g,x)), diff(f,y)-lambda*(diff(g,y)),
                    diff(f,z)-lambda*(diff(g, z)), g};
```

The `diff` command is Maple's symbolic differentiation command. You can use it for partial derivatives as here by specifying which variable you are differentiating with respect to. Then enter the following command to look for a solution of the system of Lagrange multiplier equations using Maple's numerical routine called `fsolve`:

```
S1 := fsolve(LMEqs, {x, y, z, lambda});
```

When I ran this command in Maple on my office computer, the results were (edited to show only 5 decimal places):

```
S1 := {y = 0.33333, lambda = -4.54326, x = -1.51442, z = 1.72010}
```

This should be self-explanatory.

The natural question to ask at this point is: *Is this vector (x, y, z, λ) the only solution of the Lagrange multiplier equations?* The answer is *no*! If you decide you want to look for other solutions, one way is to enter starting values for x, y, z, λ like this:

```
S2 := fsolve(LMEqs, {x=0, y=0, z=0, lambda=0});
```

This should give a second, distinct solution vector. You can think of this as saying – "OK, Maple, look for a solution of the system of equations *starting from* $(x, y, z, \lambda) = (0, 0, 0, 0)$." The solution you get will often be close to the starting point you specify if there is one. On occasion, you will just get the input command back. This means that Maple's routine failed to find a solution in a reasonable amount of time, so it stopped looking.

There is one additional solution too. See if you can find it.

Example 5

Use the techniques illustrated above to find as many distinct solutions of the Lagrange Multiplier equations for the constrained optimization problem from Problem 13 in Section 4.1 of the text as you can. (Full credit for four distinct ones, but there are quite a few more than that, and you will earn Extra Credit for each additional one you find using Maple (!))