Mathematics 241 – Multivariable Calculus
Lab Day 3 – Vector Fields and Flow Lines
October 16, 2007

Background and Goals

A *vector field* is a vector-valued function on (a subset of) $\mathbf{R}^2$ or $\mathbf{R}^3$:

$$F : X \subseteq \mathbf{R}^2 \to \mathbf{R}^2 \text{ or}$$
$$F : X \subseteq \mathbf{R}^3 \to \mathbf{R}^3.$$

We can visualize a vector field in the plane, for instance, by drawing the vector $F(x, y)$ placing its tail at the point (x, y) for all $(x, y) \in X$, or perhaps for the points in some finite subset of X such as points on a rectangular grid (this will help produce a more understandable picture!) Something similar can be done in $\mathbf{R}^3$ as well. We will use Maple today to see what a few of these look like and then investigate what are called *flow lines* of vector fields.

A worked example

We will be using two new Maple commands today that are contained in external packages. You will need to begin by entering the following commands to load these packages:

```
with(plots):
with(DEtools):
```

(The colon at the end *suppresses the output* from the command, which is a list of all the procedures in the packages. There are quite a few(!))

The command for plotting a vector field in the plane is called `fieldplot`. To generate a plot of $F(x, y) = (x - 2y, 2x + y)$ we will use the following:

```
fieldplot([x - 2*y, 2*x + y],x=-3..3,y=-3..3,arrows=SLIM,
        color=red,axes=boxed,fieldstrength=log);
```

(this should all be entered on one input line in Maple; it's broken over two lines here for typographical reasons). Notes: The x- and y- ranges specify the portion of the plane that will be plotted. The stuff after that specifies how the vectors in the vector field will be drawn, their color, the placement of the axes in the plot, and how the lengths of the arrows will be shown. Type in the command and examine the plot that is produced. Vector fields are used to represent things like:

- the velocity vectors at each point in a fluid flow, or
- the force vectors at each point due to some force (e.g. gravitational force exerted by a mass placed somewhere in space).

Imagine a fluid flowing in the plane with velocity at each point given by the vector field you just plotted. If we dropped a small, light object into the fluid at a particular point, where would the fluid carry it? For instance, try to visualize what would happen if the object started at the location $(-0.2, -0.2)$ and it "*goes with the flow*" described by the vector field.

The path followed here would be an example of what is called a *flow line* for the vector field. By definition, a flow line of a vector field F is a parametric curve $\alpha(t) = (x(t), y(t))$ such that at each point on the curve, the tangent vector to the curve *is the vector from the vector field F at that point.* In equations, this says:

$$\alpha'(t) = (x'(t), y'(t)) = F(x(t), y(t)) = (x(t) - 2y(t), 2x(t) + y(t))$$

for all t in some interval in $\mathbf{R}$. This vector equation is equivalent to the following system of *differential equations* for the x- and y-components of the flow line:

$$\begin{aligned} x'(t) &= x(t) - 2y(t) \\ y'(t) &= 2x(t) + y(t) \end{aligned}$$

We will use Maple now to see more precisely what the flow line starting at $(-0.2, -0.2)$ looks like. First enter the specification for the differential equations as above in Maple format:

```
des := [diff(x(t),t) = x(t) - 2*y(t),diff(y(t),t) = 2*x(t) + y(t)];
```

Then the following command (from the `DEtools` package loaded above) will draw the flowline (superimposed on another picture of the vector field, for comparison):

```
DEplot(des,[x(t),y(t)],t=0..4,[[x(0)=-0.2,y(0)=-0.2]],
        x=-3..3,y=-3..3,linecolor=black);
```

(Again, type this all in one one input line). Note the format of the command: First the differential equations, then `[x(t),y(t)]`, then the range of t values that will be plotted on the flow line, then the "initial conditions" – the starting point of the flow line, then the ranges of x- and y-values, then options to control the plotting. You can also plot several flow lines together if you want by including several initial conditions. For instance, try changing that part of the command above to:

```
[[x(0)=-0.2,y(0)=-0.2],[x(0) = -1, y(0) = -2]]
```

to plot two flow lines together with the vector field.

Lab Questions

A.

1) Following the worked example above, using the `fieldplot` command, plot the vector field $F(x, y) = (\sin(x+y), \cos(x-y))$ on the rectangle with $x = 0..2\pi$ (`2*Pi` in Maple), and $y = 0..2\pi$.
2) Examine your plot, and describe in words what the flow lines starting from the points $(1, 1)$, $(1, 6)$, and $(4, 3)$ appear to be doing.
3) Then, use the `DEplot` command to check your intuition about the flow lines. Did they do what you expected?

B. Repeat part A, but using the vector field $G(x, y) = (x - 3y, 3x - y)$ on the rectangle with $x = -3..3$, $y = -3..3$. In part 2, use the points $(1, 0), (-1, -1), (0, 1)$.

Maple can also plot 3D vector fields using another command called `fieldplot3d`. The format is similar to the `fieldplot` command used above, but now you must specify three component functions in the vector field, and give x-, y-, and z-ranges for the plotting.

C. Now we move up one dimension(!)

1) Let H be the vector field on $\mathbf{R}^3$ defined by

$$H(x, y, z) = \left(\frac{-x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-z}{(x^2 + y^2 + z^2)^{3/2}} \right)$$

(note this is only defined when $(x, y, z) \neq (0, 0, 0)$). Show (by hand, not using Maple), that at each point (x, y, z),] the vector $H(x, y, z)$ has magnitude inversely proportional to the square of the distance from (x, y, z) to $(0, 0, 0)$. Also show that $H(x, y, z)$ is directed back toward $(0, 0, 0)$ from the point (x, y, z). (This says H is similar to the gravitational force field exerted by a mass located at the origin).

2) Plot H using the following Maple commands:

```
r:=sqrt(x^2 + y^2 + z^2);
H:=[-x/r^3,-y/r^3,-z/r^3];
fieldplot3d(H,x=-2..2,y=-2..2,z=-2..2,fieldstrength=log,color=red,
arrows=SLIM,axes=boxed,grid=[5,5,5]);
```

3) What would the flow lines for this vector field look like?

D. Generate a 3D plot of the vector field $K(x, y, z) = (-y, x, 1)$ in the rectangular solid $x = -2..2$, $y = -2..2$, $z = -5..5$. Use the same options as in question C. What would the flow lines for this vector field look like? Careful: there are two different cases! Find an explicit parametrization of the flow line starting from $(1, 0, 0)$.