Mathematics 241 – Multivariable Calculus
Lab Day 2 – Understanding $\lim{(x, y) \to (a, b)} f(x, y)$ geometrically
September 18, 2007

Background and Goals

Last time in class we talked about what it should mean to say

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$$

for a function $f : \mathbf{R}^n \to \mathbf{R}$. The limit being L means that no matter how small we take $\varepsilon > 0$, it is possible to find some small open ball

$$B_\delta = \{\mathbf{x} \in \mathbf{R}^n \mid \|\mathbf{x} - \mathbf{a}\| < \delta\}$$

such that $f(\mathbf{x}) \in (L - \varepsilon, L + \varepsilon)$ (or equivalently, $|f(\mathbf{x}) - L| < \varepsilon$) for all $\mathbf{x} \in B_\delta$, except possibly for $\mathbf{x} = \mathbf{a}$.

Intuitively, when $n \geq 2$, we have lots of ways to let $\mathbf{x}$ approach $\mathbf{a}$ within any such open ball, and no matter how we do it, the values $f(\mathbf{x})$ should be approaching L as we get closer and closer to $\mathbf{a}$.

The goal of today's lab is to really "dig in" and understand what this means in geometric terms by analyzing a fairly complicated example where it is probably not immediately clear whether the limit exists or not. We will use Maple to draw several different kind of plots and gather evidence.

The functions in question

The functions we will study are

$$f(x, y) = \begin{cases} x \sin(1/y) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

and

$$g(x, y) = \frac{x^4 y^4}{(x^2 + y^4)^3}.$$

The questions are:

Does $\lim_{(x,y)\to(0,0)}$ of each function exist, and if so, what is it?

Lab Questions

Before starting in enter the command

```
with(plots);
```

to load the plots package.

I. In this question, you will study the graph $y = f(x, y)$ as above. Enter the following Maple command to define the function f by name:

```
f := (x,y) -> x*sin(1/y) + 1;
```

Note: We will ignore the second part of the definition in entering the function into Maple. This is OK as long as we don't end up asking Maple to evaluate f at a point with $y = 0$ to draw graphs. This is avoided with the choices of ranges of x- and y- values, and the grid options indicated.

A) (The first piece of evidence.) Use Maple to generate the graph $z = f(x, y)$, using the command

```
plot3d(f(x,y), x=-5..5,y=-5..5,grid=[100,100]);
```

Does it look like there is some z-value $z = L$ such that *all the points* on the graph for (x, y) near $(0, 0)$ are close to level L? Explain. (Suggestion: Use "Boxed" axes option and look at the scale on the z-axis.)

B) (More evidence.) To confirm or contradict your "hunch" from part A, generate new graphs by shrinking the ranges of x-and y-values as follows

1) `x = -1..1, y = -1..1`
2) `x = -0.1..0.1, y = -0.1..0.1`
3) `x = -0.01..0.01, y = -0.01..0.01`

Use "Boxed" axes option and look at the scale on the z-axis. For the smaller ranges here, you may see a $\times 10^{-3}$ printed out next to each coordinate axes. This means that the tickmarks should all be multiplied by that factor. Does this confirm or contradict what you said in part A?

II. Enter the following Maple command to define the function g by name:

```
g := (x,y) -> x^4*y^4/(x^2 + y^4)^3;
```

A) (The first piece of evidence.) Use Maple to generate the graph $z = g(x, y) = \frac{x^4 y^4}{(x^2+y^4)^3}$, using the command

```
plot3d(g(x,y), x=-5..5,y=-5..5,grid=[60,60]);
```

Does it look like there is some z-value $z = L$ such that all the points on the graph for (x, y) near $(0, 0)$ are close to level L? Explain.

B) (The second piece of evidence.) The previous question shows that the existence of a limit should be related in some way to the level curves or contours of the function g. In this part, you will draw some of them and try to analyze the $g(x, y)$ using that information. The level curve for the value c is $g(x, y) = c$, or clearing denominators

$$x^4 y^4 = c(x^2 + y^4)^3.$$

(this form is slightly more reasonable for graphing than the direct equation $g(x, y) = c$. For example, the level curve for $c = 1/4$ can be plotted in Maple using a new command called `implicitplot` like this:

```
implicitplot(x^4y^4 = (1/4)*(x^2 + y^4)^3, x=-5..5,y=-5..5,grid=[60,60]);
```

Plot several of these level curves and try to understand how they relate to the 3d plot from part A. Give the range of c values that produces nonempty level curves and describe their shapes of the level curves in words. *Note:* You can plot several of the level curves together on the same axes by putting a list of equations in [ ], separated by commas, in the `implicitplot` command. See the Maple online help feature for more details about this (and all Maple commands).

C) ("Not so fast!") In an example we looked at yesterday, we were able to tell that (for a different function), $\lim_{(x,y)\to(0,0)} g(x, y)$ *did not* exist by *letting* $(x, y) \to (0, 0)$ *along different lines through* $(0, 0)$. What happens if you try that here? (This computation can be done by hand, or by asking Maple to evaluate

```
g(t,m*t);
```

for various values of m you can supply, or even doing the same leaving m as a symbolic variable.) How does your answer here relate to the level curves you saw in part B? Explain.

D) (The final answer.) Think in more detail about your pictures from part B. What happens if you approach $(0, 0)$ *along one of those curves*? What does this say about $\lim_{(x,y)\to(0,0)} g(x, y)$.