Mathematics 241 – Multivariable Calculus
Information on Final Exam
December 3, 2007

General Information

The final exam for this course will be given at 8:30 a.m. on Tuesday, December 11. The exam will be *comprehensive*, meaning that it will cover all the material we have studied since the beginning of the semester (see list of topics to be covered below). I will write the exam to be roughly twice the length of the in-class midterms. However, you will have the full three-hour period 8:30 - 11:30 a.m. to work on the exam if you need that much time.

Review Session

I will be happy to run a daytime or evening review session to help you prepare for the final exam. We can set up a time in class on Monday, December 3.

Topics to be Covered

a) Vectors, the dot product, lengths and angles, orthogonal projections
b) The cross product, equations of lines and planes
c) Parametric curves and motion, tangent vectors and tangent lines to parametric curves, curvature
d) Vector fields, flow lines, gradient fields, curl, divergence
e) Limits and continuity for $f(x, y)$
f) Directional and partial derivatives (know the definition of $D_u f(x_0, y_0)$ and how to compute it using the definition). Differentiability for $f(x, y)$ (know the definition and what it means)
g) Derivative matrices and the general Chain Rule.
h) The gradient vector field $\nabla f(x, y)$; the geometric meaning of the vector $\nabla f(x_0, y_0)$; the relation between contours of $f(x, y)$ and flow lines of $\nabla f(x, y)$; the relation between $\nabla f(x_0, y_0)$ and directional derivatives.
i) Critical points of $f(x, y)$ and their relation with critical points of $\nabla f(x, y)$ – the "First Derivative Test"
j) The tangent plane to a graph $z = f(x, y)$.
k) The Second Derivative Test (Hessian Criterion) for critical points of $f(x, y)$. (Know the statement and how to apply it.)
l) The Lagrange Multiplier method for constrained max-min problems
m) Riemann sums and double integrals over rectangular regions in $\mathbf{R}^2$.
n) Double Integrals over non-rectangular regions via iterated integrals with variable limits of integration
o) Triple Integrals,
p) The change of variables formula for multiple integrals (know the statement and how to apply it). Polar coordinates in $\mathbf{R}^2$ and polar double integrals; cylindrical and spherical coordinates in $\mathbf{R}^3$ and cylindrical and spherical triple integrals.

q) Applications of double and triple integrals to volume, mass centers of mass, etc.
r) Line integrals and Green's Theorem (know the statement of Green's Theorem and how to apply it).

Notes

1) The only new topic not covered on the midterm exams is the last one (line integrals and Green's theorem).
2) Some of the exam questions may ask you to "put together" several of these topics.

Suggestion on How to Study

Begin by reviewing the class notes and the discussion write-ups from your group's work, but don't stop there. Look at the in-class midterm problems and try to work out solutions for those again, *especially the ones you had difficulty with the first time around*! As you do this, refer to your previous work and/or my comments on your exam sheet as needed, but be sure you can solve problems like those on the midterm exams on your own; the large majority of the final exam problems will be similar. Try a selection of the practice/review problems from the review sheets for Exams 1,2,3 again also.

Then (and only then) take a look at the practice exam questiosn and try to work out solutions:

Practice Final Exam Questions

I.

A) Find the equation of the plane containing the point $(0, 2, -1)$ and perpendicular to the line passing through the points $(0, 1, -2)$ and $(3, 2, 2)$.
B) Give a parametrization of the line from part A.
C) At what point does the line from part A intersect your plane?

II. All parts of this problem refer to the parametric curve

$$\alpha(t) = (t^2 + t - 2, t^3 + t^2 - 2t)$$

(defined for all $t \in \mathbf{R}$).

A) Thinking of $\alpha(t)$ as the position of a moving object as a function of time, at how many different times is the object at the location $(x, y) = (0, 0)$? What times are they?
B) Find all t at which the tangent vector to the curve is a horizontal vector.

III. Find the unit tangent vector and the curvature at a general point on the curve $\beta(t) = (t \cos t, t \sin t, 3t)$ in $\mathbf{R}^3$.

IV. Consider the vector field on $\mathbf{R}^3$ given by

$$\mathbf{F}(x, y, z) = (-3x, y^2 - 1, z)$$

A) Find the derivative matrix $D(\mathbf{F})$.

B) Now let $T : \mathbf{R}^3_{\rho,\phi,\theta} \to \mathbf{R}^3_{x,y,z}$ be the spherical coordinate mapping. Compute $D(F \circ T)$ two ways – once by direct substitution and differentiation, once by the Chain Rule.
C) Verify that $\alpha(t) = (e^{-3t}, \frac{e^{-t}-e^{t}}{e^{-t}+e^{t}}, e^{t})$ is a *flow line* of $\mathbf{F}$.
D) Compute the divergence and curl of $\mathbf{F}$.
E) Find a function $f : \mathbf{R}^3 \to \mathbf{R}$ such that $\nabla f = \mathbf{F}$.

V. Both parts of this question refer to $f(x, y) = x^4 - 3xy$.
A) At the point $(1, 2)$, what is the *direction of maximum rate of increase* of f? Express your result as a unit vector.
B) Find the equation of the plane in $\mathbf{R}^3$ through the point $(1, 2, f(1, 2))$, containing the vectors $(1, 0, \frac{\partial f}{\partial x}(1, 2))$ and $(0, 1, \frac{\partial f}{\partial y}(1, 2))$.

VI. The vector field plotted above in question III is the gradient vector field of the function

$$f(x, y) = y^3/3 + x^3/3 - y - x^2/2$$

A) What are the critical points of $f(x, y)$?
B) Apply the *Second Derivative Test* to each of the critical points and check your responses to part A.

VII. In the vicinity of Snowshoe Pass, which is located at coordinates $(0, 0)$, the elevation of the terrain is given by the function $f(x, y) = xy+0.9$ (x, y, f in miles). A group of hikers walks along the path given by the equation $9x^2 + (y-1)^2 = 1$, making one complete circuit of the path. What are the maximum and minimum elevations that the hikers reach?

VIII. A region R in the plane is the set of all (x, y) satisfying $x^2 + y^2 \leq 16$, $y \geq 0$, and $y \geq \sqrt{3}x$.
A) Set up iterated integral(s) to compute $\int\int_R f(x, y)\, dA$ using *rectangular coordinates*. (Use any convenient ordering of the variables: $dx\, dy$ or $dy\, dx$, then switch to the other for practice).
B) Repeat part A, but using *polar coordinates*.
C) A thin plate has the shape of R and density $d(x, y) = y^2$ at the point (x, y). Compute its *center of mass*.

IX. A hole is drilled through a solid sphere of radius 3 centered at the origin in $\mathbf{R}^3$, removing all the material that was at points satisfying $x^2 + y^2 \leq 1$. What is the volume of the remaining solid?

X. Let $\mathbf{F}$ be the vector field $(P(x, y), Q(x, y)) = (x^2 + 3y, y^2 - x)$.
A) Let $\mathbf{x}(t) = (t, t^2)$, $0 \leq t \leq 1$. Compute the line integral $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{x}$.
B) State Green's Theorem.
C) Let D be the closed disk $D = \{(x, y) : (x-1)^2 + y^2 \leq 1\}$. Verify that the conclusion of Green's Theorem holds for this region and this vector field.