Mathematics 241 – Multivariable Calculus
Solutions for Exam 3 Review Problems
November 28, 2007

I. Let $\mathbf{F}(x, y, z) = (x + yz, y + xz, z + xy)$ be a vector field on $\mathbf{R}^3$.

A) Compute the divergence of $\mathbf{F}$.

Solution: The divergence of a vector field $\mathbf{F}$ is the scalar-valued function $\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$. Here:

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x + yz, y + xz, z + xy) = 3.$$

B) Compute the curl of $\mathbf{F}$.

Solution: The curl of a vector field $\mathbf{F}$ is the vector field $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$ (cross product). Here:

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (x + yz, y + xz, z + xy) = (0, 0, 0).$$

C) Is there a scalar-valued function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$? Why or why not?

Solution: For a scalar-valued function, ∇f is the gradient vector field. There are such functions:

$$f = \frac{1}{2}(x^2 + y^2 + z^2) + xyz + C$$

where C is an arbitrary constant.

II. All parts of this question refer to the function

$$f(x, y) = xe^{-x^2} \cos(y).$$

A) Compute the second degree Taylor polynomial of f at $(a, b) = (0, 0)$. Give both the form with the derivative and Hessian matrices and the expanded ("messy, practical") form.

Solution: The form with the derivative and Hessian matrices is

$$p_2(x, y) = f(0, 0) + D(f)(0, 0) \begin{pmatrix} x - 0 \\ y - 0 \end{pmatrix} + \frac{1}{2}(x - 0 \ \ y - 0) D^2(f) \begin{pmatrix} x - 0 \\ y - 0 \end{pmatrix}.$$

Expanded out this becomes

$$f(0,0) + \frac{\partial f}{\partial x}(0,0)(x - 0) + \frac{\partial f}{\partial x}(0,0)(x - 0)$$
$$+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(0,0)(x - 0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(0,0)(x - 0)(y - 0) + \frac{\partial^2 f}{\partial y^2}(0,0)(y - 0)^2\right).$$

For our function $f(x,y)$,

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \cos(y)e^{-x^2}(1-2x^2) \Rightarrow \frac{\partial f}{\partial x}(0,0) = 1 \\
\frac{\partial f}{\partial y} &= -\sin(y)xe^{-x^2} \Rightarrow \frac{\partial f}{\partial x}(0,0) = 0 \\
\frac{\partial^2 f}{\partial x^2} &= \cos(y)e^{-x^2}(-6x+4x^3) \Rightarrow \frac{\partial^2 f}{\partial x^2}(0,0) = 0 \\
\frac{\partial^2 f}{\partial x \partial y} &= -\sin(y)e^{-x^2}(1-2x^2) \Rightarrow \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0 \\
\frac{\partial^2 f}{\partial y^2} &= -\cos(y)xe^{-x^2} \Rightarrow \frac{\partial f}{\partial x}(0,0) = 0
\end{aligned}$$

Hence

$$p_2(x,y) = x.$$

B) $f(x,y)$ has exactly three critical points with $0 \le y \le 2\pi/3$. Find them.

Solution: The critical points are the simultaneous solutions of $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. From the computations of the partial derivatives in part A, for y in the range given,

$$\frac{\partial f}{\partial x} = \cos(y)e^{-x^2}(1-2x^2) = 0$$

implies $y = \pi/2$ or $x = \pm 1/\sqrt{2}$. Then

$$\frac{\partial f}{\partial y} = -\sin(y)xe^{-x^2} = 0$$

implies $y = 0$ or $x = 0$. Putting these together, we find the three critical points:

$$(0, \pi/2),\ (1/\sqrt{2}, 0),\ (-1/\sqrt{2}, 0).$$

C) Classify the critical points from part B as local maxima, local minima, or saddle points using the Second Derivative Test (Hessian Criterion).

Solution: Using the partial derivatives computed in part A,

$$D^2(f) = \begin{pmatrix} \cos(y)e^{-x^2}(-6x+4x^3) & -\sin(y)e^{-x^2}(1-2x^2) \\ -\sin(y)e^{-x^2}(1-2x^2) & -\cos(y)xe^{-x^2} \end{pmatrix}$$

- At $(0, \pi/2)$,

$$D^2(f)(0,\pi/2) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

The determinant is $\det(D^2(f)) = -1$, so this is a *saddle point.*

- At $(1/\sqrt{2}, 0)$,

$$D^2(f)(1/\sqrt{2}, 0) = \begin{pmatrix} -4e^{-1/2}/\sqrt{2} & 0 \\ 0 & -e^{-1/2}/\sqrt{2} \end{pmatrix}.$$

  The upper left entry is negative and the determinant is $\det(D^2(f)) = 2e^{-1} > 0$, so this is a *local maximum.*

- At $(-1/\sqrt{2}, 0)$,

$$D^2(f)(-1/\sqrt{2}, 0) = \begin{pmatrix} 4e^{-1/2}/\sqrt{2} & 0 \\ 0 & e^{-1/2}/\sqrt{2} \end{pmatrix}.$$

  The upper left entry is positive and the determinant is $\det(D^2(f)) = 2e^{-1} > 0$, so this is a *local minimum.*

III. Let $f(x, y) = (x-1)^2 + y^2$ and $g(x, y) = x^2/4 + y^2$.

A) Find the largest and smallest values of f on the constraint curve defined by $g(x, y) = 1$ using the method of Lagrange Multipliers.

Solution: The Lagrange multiplier equations are:

$$\begin{aligned} 2(x-1) &= \lambda x/2 \\ 2y &= 2\lambda y \\ x^2/4 + y^2 - 1 &= 0. \end{aligned}$$

From the second equation, $y = 0$ or $\lambda = 1$. In the first case, from the constraint equation, $x = \pm 2$. In the second case, from the first equation $2x - 2 = x/2$, so $x = 4/3$ and then $y = \pm\sqrt{5}/3$ from the constraint equation. To decide which points are maxima and which are minima of f on the constraint curve we substitute:

$$\begin{aligned} f(2, 0) &= 1 \\ f(-2, 0) &= 9 \\ f(4/3, \pm\sqrt{5}/3) &= 2/3 \end{aligned}$$

So the maximum value is 9 and the minimum value is 2/3.

B) Sketch the level curves of f for the values $c = 2/3, 1, 9$ together with the curve $g(x, y) = 1$ and explain how your sketch relates to the result of part A.

Solution: The level curves of f for $c > 0$ are circles with center at $(1, 0)$ and radius $\sqrt{c}$. The constraint curve $g(x, y) = 1$ is an ellipse with semimajor axis 2 along the x-axis and semiminor axis 1 along the y-axis. The level curves of f for the values $c = 2/3, 1, 9$ are tangent to the ellipse at the constrained critical points found in part A.

C) What are the absolute maximum and minimum values of $h(x,y) < /EM >= 10 - f(x,y)$ on the region defined by $g(x,y) \leq 1$?

Solution: $h(x,y) = 10 - (x-1)^2 - y^2$ has a critical point at $(x,y) = (1,0)$ which is a local and global maximum. Applying Lagrange multipliers to h on the constraint curve gives a system of equations with the same solutions as those from part A. We have $h(2,0) = 9$, $h(-2,0) = 1$, and $h(4/3, \pm\sqrt{5}/3) = 28/3$. So the maximum of $h(x,y)$ is 10 and the minimum is 1.

IV. All parts of this question refer to the region R in the plane *inside* the circle $x^2 + y^2 = 5$, *below* parabola $y = x^2 + 1$, and above the x-axis.

A) Set up iterated integral(s) to integrate a general function $f(x,y)$ over R, integrating first with respect to x.

Solution: The parabola crosses the circle at $(\pm 1, 2)$ and extends only down to $y = 1$. So to integrate first with respect to x (type 2 regions), we actually need to subdivide D into three regions: two with $1 \leq y \leq 2$ and one with $0 \leq y \leq 1$:

$$\int_1^2 \int_{-\sqrt{5-y^2}}^{-\sqrt{y-1}} f(x,y)\ dx\ dy + \int_1^2 \int_{\sqrt{5-y^2}}^{\sqrt{y-1}} f(x,y)\ dx\ dy + \int_0^1 \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} f(x,y)\ dx\ dy.$$

B) Same question as A, but interchanging the order of integration (integrating first with respect to y).

Solution: The setup using type 1 regions is similar since the top edge of the region changes at $x = -1$ and again at $x = 1$:

$$\int_{-\sqrt{5}}^{-1} \int_0^{\sqrt{5-x^2}} f(x,y)\ dy\ dx + \int_{-1}^{1} \int_0^{x^2+1} f(x,y)\ dy\ dx + \int_1^{\sqrt{5}} \int_0^{\sqrt{5-x^2}} f(x,y)\ dy\ dx.$$

V.

A) State the change of variables formula for double integrals in general.

Solution: See the text and the class notes. The key points are that if $T : \mathbf{R}^2_{u,v} \to \mathbf{R}^2_{x,y}$ is a change of coordinate mapping, and $T(D*) = D$, then

$$\iint_D f(x,y)\ dxdy = \iint_{D*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|\ dudv.$$

Here

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

is the Jacobian determinant – the determinant of the derivative matrix of the mapping T.

B) Where does the "extra r" in the polar coordinate integral $\int\int f(r\cos\theta, r\sin\theta)r\ dr\ d\theta$ come from?

Solution: It comes from the Jacobian determinant of the polar coordinate mapping $T(r,\theta) = (r\cos\theta, r\sin\theta)$:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

C) Change variables to polar coordinates to determine the integral of $f(x,y) = xy$ over the quarter circle inside $x^2 + y^2 = 4$ in the *second* quadrant.

Solution: This is

$$\begin{aligned}\int\int_D xy\ dxdy &= \int_{\pi/2}^{\pi}\int_0^2 (r\cos\theta)(r\sin\theta)\ rdrd\theta \\ &= \int_{\pi/2}^{\pi}\int_0^2 r^3\cos\theta\sin\theta\ drd\theta \\ &= \frac{r^4}{4}\Big|_0^2 \cdot \frac{1}{2}\sin^2\theta\Big|_{\pi/2}^{\pi} \\ &= -2.\end{aligned}$$

D) Change variables to spherical coordinates in $\mathbf{R}^3$ to evaluate the triple integral of the function x over the portion of the solid ball of radius 1 with center at (0,0,0) in the first octant (i.e. the part of the region inside the sphere of radius 1, with x, y, z all ≥ 0).

Solution: This is

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho\sin\phi\cos\theta \cdot \rho^2\sin\phi\ d\rho\, d\phi\, d\theta = \frac{\pi}{16}.$$

VI. Find the volume of the solid bounded above by the cone $z = (x^2+y^2)^{1/2}$, below by $z = 0$, and by $0 \leq x^2 + y^2 \leq 4$. You may use any convenient coordinate system to carry out the necessary computations.

Solution: Because of the circular symmetry of the region about the z-axis, *cylindrical coordinates* are indicated for this one. The volume is

$$V = \int_0^{2\pi}\int_0^2\int_0^r 1\ rdz\, dr\, d\theta = \frac{16\pi}{3}.$$