Mathematics 241 – Multivariable Calculus
Solutions for Midterm Exam 1
September 28, 2007

I.

A) A direction vector for the line is $Q - P = (1,2,-4) - (0,0,1) = (1,2,-5)$. Parametric equations come from $P + t(Q-P) = (0,0,1) + t(1,2,-5)$, so

$$\begin{aligned} x &= t \\ y &= 2t \\ z &= 1 - 5t. \end{aligned}$$

The interval of t-values giving the line segment from P to Q for these parametric equations is $0 \le t \le 1$ (note that $t = 0$ gives $(x,y,z) = (0,0,1)$ which is P, and $t = 1$ gives $(1,2,-4)$, which is Q). The point equidistant from P and Q along the line is the midpoint of the line segment, which is the point with $t = \frac{1}{2}$. This can be seen in several different ways. The easiest is just to notice that the parametric equations correspond to motion with constant speed along the line segment, so $t = \frac{1}{2}$ will take us exactly half the distance from P to Q.

B. Let $v = Q - P = (1,2,-5)$ as above. Also let $w = R - P = (1,-1,0) - (0,0,1) = (1,-1,-1)$. Then a normal vector for the desired plane is

$$\mathbf{N} = (1,2,-5) \times (1,-1,-1) = (-2-5, -(-1+5), -1-2) = (-7,-4,-3).$$

The Cartesian equation of the plane is

$$0 = \mathbf{N} \cdot (x-0, y-0, z-1) = (-7,-4,-3) \cdot (x,y,z-1) = -7x - 4y - 3z + 3,$$

or $7x + 4y + 3z = 3$.

C. The desired angle is

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{v \cdot w}{\|v\|\|w\|}\right) \\ &= \cos^{-1}\left(\frac{4}{\sqrt{30}\sqrt{3}}\right) \\ &\doteq 1.13 \text{ radians, or} \\ &\doteq 65.1^\circ. \end{aligned}$$

II. Note that $(x,y,z) \cdot (0,0,1) = z$, and $\|(x,y,z)\| = \sqrt{x^2+y^2+z^2}$. Hence the equation given can be rewritten as

$$\frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{1}{\sqrt{2}}.$$

Changing to spherical coordinates, we know $z = \rho\cos(\varphi)$, and $\rho^2 = x^2+y^2+z^2$. Hence the equation can be rewritten as

$$\frac{1}{\sqrt{2}} = \frac{\rho\cos(\varphi)}{\rho} = \cos(\varphi).$$

For $0 \le \varphi \le \pi$, this has just one solution, $\varphi = \frac{\pi}{4}$. The set of all points satisfying this equation (equivalent to the original one) is a *cone* with axis along the z-axis.

Comment: A common error on this problem was to take the equation $\frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{1}{\sqrt{2}}$ and equate numerators and denominators to obtain $z = 1$ and $\sqrt{x^2+y^2+z^2} = 2$. *This is NOT a valid algebraic maneuver.* Knowing $\frac{a}{b} = \frac{c}{d}$ does not imply that $a = c$ and $b = d$; it just implies that $ad = bc$.

III.

A. Letting (x, y) go to $(0, 0)$ along the line $y = mx$,

$$\begin{aligned}
\lim_{x\to 0} f(x, mx) &= \lim_{x\to 0} \frac{x \cdot mx - (mx)^2}{x^2 + 4(mx)^2} \\
&= \lim_{x\to 0} \frac{x^2(m - m^2)}{x^2(1 + 4m^2)} \\
&= \lim_{x\to 0} \frac{m - m^2}{1 + 4m^2} \text{ (cancel the } x^2\text{'s)} \\
&= \frac{m - m^2}{1 + 4m^2}
\end{aligned}$$

B. From the computations in part A, we see that $f(x, mx)$ is approaching different values as $x \to 0$, depending on the value of m. For example, we get a limiting value of 0 if $m = 1$, but a limiting value of $-\frac{2}{5}$ if $m = -1$. This implies that $\lim_{(x,y)\to(0,0)} f(x, y)$ *does not exist.* Hence f cannot be continuous at $(0, 0)$, since the definition of continuity is that $\lim_{(x,y)\to(0,0)} f(x, y)$ must exist and equal $f(0, 0)$.

IV.

A. The Cartesian equation of the graph is $z = \cos(\sqrt{x^2 + y^2})$. In cylindrical coordinates, $r^2 = x^2 + y^2$ and z is the Cartesian z-coordinate, so the cylindrical equation is just $z = \cos(r)$.

B. The equation of the level curve is $\cos(r) = 1$. This implies $r = 2n\pi$ some integer $n \ge 0$. If $n = 0$, we get a single point at the origin. If $n \ge 1$, then the polar equation $r = 2n\pi$ defines a circle of radius $2n\pi$ in the polar coordinate plane. Thus, the level curve consists of an isolated point at the origin together with an infinite collection of circles centered at the origin, with radii $2n\pi$ for $n = 1, 2, 3, \ldots$.

Comment: Of course you can get the same conclusion by going back to the Cartesian coordinate equation of the graph. The set of points in the plane with $\sqrt{x^2+y^2} = 2n\pi$ is the circle described above when $n \geq 1$.

C. By the chain rule for 1-variable derivatives:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\cos(\sqrt{x^2+y^2})\right) = -\sin(\sqrt{x^2+y^2}) \cdot \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x.$$

This can be simplified to give

$$\frac{\partial f}{\partial x} = \frac{-x\sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}.$$

Substituting $x = \frac{\pi}{2}$, we obtain

$$\frac{\partial f}{\partial x}\left(\frac{\pi}{2}, 0\right) = \frac{-\frac{\pi}{2}\sin(\frac{\pi}{2})}{\frac{\pi}{2}} = -1.$$