Mathematics 241 – Multivariable Calculus
Sample Exam Questions for Exam 2 – Solutions
October 20, 2007

I.

A) By direct computation,

$$\frac{\partial w}{\partial s} = -t\sin^2(st) + t\cos^2(st) \quad \text{and}$$
$$\frac{\partial w}{\partial t} = -s\sin^2(st) + s\cos^2(st)$$

Hence

$$s\frac{\partial w}{\partial s} = -st\sin^2(st) + st\cos^2(st) = t\frac{\partial w}{\partial t}.$$

B) By the Chain Rule,

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s} = f'(st)t$$

and

$$\frac{\partial w}{\partial t} = \frac{dw}{dx}\frac{\partial x}{\partial t} = f'(st)s$$

Hence

$$s\frac{\partial w}{\partial s} = stf'(st) = t\frac{\partial w}{\partial t}.$$

II.

A) We have $f_x = \frac{\partial f}{\partial x} = e^{x-y}$ and $f_y = \frac{\partial f}{\partial x} = -e^{x-y}$. These functions are continuous at all $(x, y) \in \mathbf{R}^2$. Hence f is differentiable at all $(x, y) \in \mathbf{R}^2$. (See Theorem 3.5 on page 115 of the text.)

B) By the definition,

$$D(f) = (\, f_x \quad f_y \,) = (\, e^{x-y} \quad -e^{x-y} \,)$$

and

$$D(g) = \begin{pmatrix} (g_1)_s & (g_1)_t \\ (g_2)_s & (g_2)_t \end{pmatrix} = \begin{pmatrix} 2s & -2t \\ t & s+2 \end{pmatrix}.$$

C) The tangent plane is given by

$$z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) = 1 + (x-1) - (y-1) = 1 + x - y.$$

This meets the z-axis when $(x, y) = (0, 0)$, and then $z = 1$. So the intersection point is $(0, 0, 1)$.

D) The directional derivative is given as follows. Let $u = v/\|v\| = (1/\sqrt{10}, 3/\sqrt{10})$ be the unit vector in the direction of v. So then

$$\begin{aligned} D_u f(2,1) &= \nabla f(2,1) \cdot (1/\sqrt{10}, 3/\sqrt{10}) \\ &= (f_x(2,1), f_y(2,1)) \cdot (1/\sqrt{10}, 3/\sqrt{10}) \\ &= (e, -e) \cdot (1/\sqrt{10}, 3/\sqrt{10}) \\ &= -2e/\sqrt{10}. \end{aligned}$$

E) By the Chain Rule, since $g(3,0) = (9,0)$,

$$\begin{aligned}
D(f \circ g)(3,0) &= D(f)(g(3,0))D(g)(3,0) \\
&= \begin{pmatrix} e^{x-y} & -e^{x-y} \end{pmatrix} |_{(x,y)=(9,0)} \begin{pmatrix} 2s & -2t \\ t & s+2 \end{pmatrix} |_{(s,t)=(3,0)} \\
&= \begin{pmatrix} e^9 & -e^9 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix} \\
&= \begin{pmatrix} 6e^9 & -5e^9 \end{pmatrix}
\end{aligned}$$

F) Substituting and computing the derivatives directly,

$$(f \circ g)(s,t) = e^{s^2-t^2-st-2t}$$

So

$$D(f \circ g)(s,t) = \begin{pmatrix} (2s-t)e^{s^2-t^2-st-2t} & (-2t-s-2)e^{s^2-t^2-st-2t} \end{pmatrix}$$

Hence substituting $(s,t) = (3,0)$,

$$D(f \circ g)(3,0) = \begin{pmatrix} 6e^9 & -5e^9 \end{pmatrix},$$

which is the same as above.

III.

A) By our general knowledge of directional derivatives, f is increasing the fastest at $(1,1)$ in the direction of $\nabla f(1,1)$. We have $\nabla f(x,y) = (-2ax, -2by)$ so this means we want a unit vector in the direction of $(-2a, -2b)$:

$$\mathbf{u} = \left(\frac{-a}{\sqrt{a^2+b^2}}, \frac{-b}{\sqrt{a^2+b^2}} \right).$$

B) The marble would roll in the direction that decreased the altitude the fastest, hence in the direction of $-\nabla f(1,1) = (2a, 2b)$, so in direction of

$$\mathbf{v} = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right).$$

C) To remain at a constant altitude, move in a direction perpendicular to $\nabla f(1,1)$, so in either of the directions specified by

$$\mathbf{w} = \pm \left(\frac{b}{\sqrt{a^2+b^2}}, \frac{-a}{\sqrt{a^2+b^2}} \right)$$

(These are chosen to make $\mathbf{w} \cdot \mathbf{u} = 0$ using the $\mathbf{u}$ from part A.)

IV. Apply the Chain Rule, thinking of the polar coordinates map $x = r\cos\theta$, $y = r\sin\theta$ as the "inside function." We have

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\
&= \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \quad \text{and} \\
\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} \\
&= -\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta
\end{aligned}$$

Hence starting from the right side of the desired equation,

$$\begin{aligned}
\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 &= \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 + \frac{1}{r^2}\left(-\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta\right)^2 \\
&= \left(\frac{\partial z}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}(\cos\theta\sin\theta - \cos\theta\sin\theta) \\
&\quad + \left(\frac{\partial z}{\partial y}\right)^2(\cos^2\theta + \sin^2\theta) \\
&= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2,
\end{aligned}$$

which is what we wanted to show(!)

V.

A) We have $x'(t) = (3\cos(3t), -3\sin(3t), 3t^{1/2})$, so

$$\|x'(t)\| = 3\sqrt{\cos^2(3t) + \sin^2(3t) + t} = 3\sqrt{1+t}.$$

Hence the desired arclength is

$$L = \int_0^1 3\sqrt{1+t}\,dt = 2(1+t)^{3/2}|_0^1 = 2(2\sqrt{2} - 1) = 4\sqrt{2} - 2.$$

B) The unit tangent vector is

$$\begin{aligned}
\mathbf{T} &= \frac{1}{\|x'(t)\|}x'(t) = \frac{1}{3\sqrt{1+t}}(3\cos(3t), -3\sin(3t), 3t^{1/2}) \\
&= \left(\frac{\cos(3t)}{\sqrt{1+t}}, \frac{-\sin(3t)}{\sqrt{1+t}}, \sqrt{\frac{t}{1+t}}\right).
\end{aligned}$$

C) $x'(t)$ is given above in part A. Then

$$x''(t) = (-9\sin(3t), -9\cos(3t), \frac{3}{2}t^{-1/2}).$$

Using the alternate curvature formula, we compute

$$x'(t) \times x''(t) = \Big(-\frac{9\sin(3t)}{2\sqrt{t}} + 27\sqrt{t}\cos(3t), -27\sqrt{t}\sin(3t) - \frac{9\cos(3t)}{2\sqrt{t}}, \\ -27(\cos^2(3t) - \sin^2(3t)) \Big)$$

Hence $\kappa(t) = \|x'(t) \times x''(t)\| / \|x'(t)\|^3 =$

$$\frac{81}{4}\frac{36t^2 + 36t + 1}{t} \cdot \frac{1}{(3\sqrt{1+t})^3} = \frac{3}{4}\frac{36t^2 + 36t + 1}{(1+t)\sqrt{1+t}}$$

VI.

A) This is clear – if any two of x, y, z are zero, then all the component functions of $\mathbf{F}(x, y, z) = (yz, xz, xy)$ are zero.

B) This will not be covered on the exam (as announced on Friday, 10/19).

C) This will not be covered on the exam (as announced on Friday, 10/19).

D) $f = xyz + C$ for any constant.

E) The curve given by $x = y = z = \frac{1}{1-t}$ is a flow line of $\mathbf{F}$ since

$$x'(t) = y'(t) = z'(t) = (-1)(1-t)^{-2}(-1) = \frac{1}{(1-t)^2} = y(t)z(t) = x(t)z(t) = x(t)y(t).$$