I. In this problem, \( P = (1, 0, 1), \) \( Q = (-2, 3, 2) \), and \( R = (1, 2, 0) \).

A)  (10) Find the equation of the plane containing the points \( P, Q, R \) in \( \mathbb{R}^3 \).

**Solution:** The displacement vector from \( P \) to \( Q \) is \( \mathbf{v} = Q - P = (-3, 3, 1) \) and the vector from \( P \) to \( R \) is \( \mathbf{w} = R - P = (0, 2, -1) \). For the plane we can take \( N = ( -3, 3, 1 ) \times ( 0, 2, -1 ) = ( -5, -3, -6 ) \). Then the equation of the plane is \( 0 = N \cdot ( x - 1, y - 0, z - 1 ) = -5x + 5 - 3y - 6z + 6 \), or \( 5x + 3y + 6z = 11 \).

B)  (10) At what point does the line containing \( P, Q \) meet the \( xy \)-plane?

**Solution:** The line is \( (1, 0, 1) + (-3, 3, 1)t = (1 - 3t, 3t, 1 + t) \). This meets the \( xy \)-plane when \( z = 1 + t = 0 \), so \( t = -1 \). The point of intersection is \( (4, -3, 0) \).

C)  (5) If \( \mathbf{v} \) is the displacement vector from \( P \) to \( Q \) and \( \mathbf{w} \) is the displacement vector from \( P \) to \( R \), at what angle do \( \mathbf{v}, \mathbf{w} \) meet?

**Solution:** The angle \( \theta \) satisfies \( \cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\| \mathbf{v} \| \| \mathbf{w} \|} = \frac{5}{\sqrt{19}} \). So

\[
\theta = \cos^{-1} \left( \frac{\sqrt{5/19}}{} \right) \approx 1.032 \text{ radians}.
\]

II.

A)  (7.5) Let \( F : \mathbb{R}^2_{u,v} \rightarrow \mathbb{R}^2_{x,y} \), \( F(u, v) = (x(u, v), y(u, v)) = (u^2 - v^2, 2uv) \) and let \( g : \mathbb{R}^2_{x,y} \rightarrow \mathbb{R} \), \( g(x, y) = \sin(x) \cos(y) \). Find the derivative matrix \( D(g \circ F) \) by direct substitution and differentiation.

**Solution:** The function \((g \circ F)(u, v) = \sin(u^2 - v^2) \cos(2uv)\), so computing partial derivatives by the product rule

\[
\begin{align*}
\frac{\partial (g \circ F)}{\partial u} &= 2u \cos(u^2 - v^2) \cos(2uv) - 2v \sin(u^2 - v^2) \sin(2uv) \\
\frac{\partial (g \circ F)}{\partial u} &= -2v \cos(u^2 - v^2) \cos(2uv) - 2u \sin(u^2 - v^2) \sin(2uv)
\end{align*}
\]

and the derivative matrix is the \( 1 \times 2 \) matrix with these entries.

B)  (7.5) Compute \( D(g \circ F) \) for the functions in part A using the Chain Rule and show you get the same result as in part A.

**Solution:** The Chain Rule says \( D(g \circ F) = D(g)(F(u, v)) D(F)(u, v) \) (matrix product). So we compute \( D(g) = \begin{pmatrix} \cos(x) \cos(y) & -\sin(x) \sin(y) \end{pmatrix} \) and

\[
D(F) = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}.
\]

Hence the product matrix to be computed is

\[
\begin{pmatrix} \cos(u^2 - v^2) \cos(2uv) & -\sin(u^2 - v^2) \sin(2uv) \end{pmatrix} \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}
\]
which gives a $1 \times 2$ matrix with first entry

$$2u \cos(u^2 - v^2) \cos(2uv) - 2v \sin(u^2 - v^2) \sin(2uv)$$

and second entry

$$-2v \cos(u^2 - v^2) \cos(2uv) - 2u \sin(u^2 - v^2) \sin(2uv)$$

as in the matrix $D(g \circ F)$.

C) (10) Now let $z = g(x, y)$ for a general function $g$ — not specifically $g(x, y)$ as above in part A, and $x = u^2 - v^2, y = 2uv$ as in the definition of $F$ above. Show that

$$\frac{1}{4(u^2 + v^2)} ((z_u)^2 + (z_v)^2) = (z_x)^2 + (z_y)^2.$$  

**Solution:** Now we use the expanded form of the Chain Rule: We have $z_u = z_x x_u + z_y y_u$ and $z_v = z_x x_v + z_y y_v$ so

$$(z_u)^2 + (z_v)^2 = (z_xx_u + z_y y_u)^2 + (z_xx_v + z_y y_v)^2 = (2uz_x + 2vz_y)^2 + (-2vz_x + 2uz_y)^2 = 4u^2(z_x)^2 + 8uvz_xz_y + 4v^2(z_y)^2 + 4v^2(z_x)^2 - 8uvz_xz_y + 4v^2(z_y)^2 = 4(u^2 + v^2)((z_x)^2 + (z_y)^2)$$

Dividing by $4(u^2 + v^2)$ yields the desired equation.

III. All parts of this problem refer to $f(x, y) = (x + 1)^2 + y^2$.

A) (5) Sketch the level curves of $f(x, y)$ for the values $c = 1, 4, 9$.

**Solution:** The level curves of $f$ are circles with center at $(-1, 0)$ the radii are $r = 1, 2, 3$ respectively.

B) (10) At the point $(1, 2)$, in which direction is $f$ increasing the fastest? Express your answer as a unit direction vector.

**Solution:** This is in the direction of the gradient vector $\nabla f(1, 2)$. The gradient vector is $\nabla f(x, y) = (2(x + 1), 2y)$ at a general point. So $\nabla f(1, 2) = (4, 4)$. The unit vector in this direction is $\frac{1}{4\sqrt{2}}(4, 4) = (\sqrt{2}/2, \sqrt{2}/2)$.

C) (15) Find the points on the curve $g(x, y) = x^2 + y^2 = 1$ at which $f(x, y)$ takes its largest and smallest values. What is true about the vectors $\nabla f$ and $\nabla g$ at your points?

**Solution:** Using the Lagrange multiplier method, we must solve

$$2(x + 1) = \lambda x/2$$

$$2y = 2\lambda y$$

$$\frac{x^2}{4} + y^2 = 1$$
From the second equation, $y = 0$ or $\lambda = 1$. If $y = 0$, the constraint equation gives $x = \pm 2$, so we obtain two points $(\pm 2, 0)$. If $\lambda = 1$, then from the first equation, $2(x + 1) = x/2$, so $x = -4/3$. Then from the constraint equation we get $y = \pm \sqrt{5}/3$. To determine which of these give maximum and minimum values, we substitute into $f(x, y)$:

$$f(2, 0) = 9 \text{ (maximum)}$$
$$f(-2, 0) = 1$$
$$f(-4/3, \pm \sqrt{5}/3) = 1/9 + 5/9 = 2/3 \text{ (minimum)}$$

The points we found here are the points where the level curve of $f$ passing through that point and the constraint curve are tangent.

IV. Let $f(x, y) = xe^{-2x^2-y^2}$.

A) (10) Find the equation of the tangent plane to the graph $z = f(x, y)$ at the point $(1, 1, e^{-3})$.

**Solution:** We must compute the partial derivatives to start:

$$f_x = (1 - 4x^2)e^{-2x^2-y^2}$$
$$f_y = -2xye^{-2x^2-y^2}.$$

At $(x, y) = (1, 1)$, $f_x(1, 1) = -3e^{-3}$, and $f_y(1, 1) = -2e^{-3}$, so the tangent plane is

$$z = e^{-3} - 3e^{-3}(x - 1) - 2e^{-3}(y - 1).$$

B) (10) Find all the critical points of $f(x, y)$.

**Solution:** The critical points are the solutions of $f_x = 0$ and $f_y = 0$. Using the formulas for $f_x$, $f_y$ from part A, we see that $f_x = 0$ when $x = \pm 1/2$ and $f_y = 0$ when $x = 0$ or $y = 0$ (Note: the exponential factor is never zero.) Hence the simultaneous solutions are the two points $(\pm 1/2, 0)$.

C) (20) Use the second derivative test (Hessian criterion) to determine the type of each critical point you found in part B.

**Solution:** Now we need the second-order partial derivatives as well:

$$f_{xx} = (16x^3 - 12x)e^{-2x^2-y^2}$$
$$f_{xy} = (1 - 4x^2)(-2y)e^{-2x^2-y^2}$$
$$f_{yy} = -2x(1 - 2y^2)e^{-2x^2-y^2}$$

So at $(1/2, 0)$ the Hessian matrix is

$$D^2(f)(1/2, 0) = \begin{pmatrix} -4e^{-1/2} & 0 \\ 0 & -e^{-1/2} \end{pmatrix}$$
The determinant is $4e^{-1} > 0$ and the upper left entry is $< 0$ so this is a \textit{local maximum}. At $(-1/2, 0)$ the Hessian matrix is

$$D^2(f)(-1/2, 0) = \begin{pmatrix} 4e^{-1/2} & 0 \\ 0 & e^{-1/2} \end{pmatrix}$$

The determinant is $4e^{-1} > 0$ and the upper left entry is $> 0$ so this is a \textit{local minimum}.

V. A region $R$ in $\mathbb{R}^2$ is the set of points satisfying $x^2 + y^2 \geq 1$, $y \geq x$, $x \geq 0$, and $y \leq 4$.

A) (5) Sketch the region $R$.

\textbf{Solution:} This is the region outside the unit circle with center $(0, 0)$, to the right of the $y$-axis, below the horizontal line $y = 4$, and above the line $y = x$.

B) (10) Set up the limits of integration of iterated integral(s) to compute $\int \int_R f(x, y) \, dA$ integrating with respect to $x$ first, then $y$.

\textbf{Solution:} The circle intersects the line $y = x$ at $(\sqrt{2}/2, \sqrt{2}/2)$. From there to the top of the circle at $y = 1$, the left boundary of the region is part of the circle. For $y > 1$, though, the left boundary is part of the $y$-axis so we have to split the integral at $y = 1$:

$$\int_{1}^{\sqrt{2}/2} \int_{\sqrt{1-y^2}}^{y} f(x, y) \, dx \, dy + \int_{1}^{4} \int_{0}^{y} f(x, y) \, dx \, dy.$$ 

C) (10) Now reverse the order of the variables and set up iterated integral(s) to compute the same integral, but integrating with respect to $y$ first, then $x$.

\textbf{Solution:} We also need to split the integral this way since the bottom boundary changes at $x = \sqrt{2}/2$. The region extends all the way to $x = 4$ on the right, where the line $y = 4$ intersects $y = x$:

$$\int_{0}^{\sqrt{2}/2} \int_{x}^{4} f(x, y) \, dy \, dx + \int_{\sqrt{2}/2}^{4} \int_{x}^{4} f(x, y) \, dy \, dx.$$ 

VI. (20) The metal making up a solid slug having the shape of the region in $\mathbb{R}^3$ with $x^2 + y^2 \leq 4$ and $-1 \leq z \leq 1$ has density $\delta(x, y, z) = 5 + x$ at all points. Determine the coordinates of its center of mass.

\textbf{Solution:} We will set up the triple integrals to compute the coordinates of the center of mass using \textit{cylindrical} coordinates, since the slug is just a cylinder with axis along
the $z$-axis. The total mass is
\[
M = \int_0^{2\pi} \int_0^2 \int_{-1}^1 (5 + r \cos \theta)rdz \ dr \ d\theta \\
= 2 \int_0^{2\pi} \int_0^2 5r + r^2 \cos \theta \ dr \ d\theta \\
= 2 \int_0^{2\pi} \left. \frac{5}{2} r^2 + \frac{1}{3} r^3 \cos \theta \right|_0^2 \ d\theta \\
= 2 \int_0^{2\pi} \left. 10 + \frac{8}{3} \cos \theta \right|_0^2 \ d\theta \\
= 20\theta + \frac{8}{2} \sin \theta \bigg|_0^{2\pi} \\
= 40\pi.
\]
Since the density does not depend on $y$ or $z$, by the symmetry of the cylinder, it can be seen that $\bar{y} = \bar{z} = 0$. To compute the $x$-coordinate,
\[
\bar{x} = \frac{1}{40\pi} \int_0^{2\pi} \int_0^2 \int_{-1}^1 r \cos \theta (5 + r \cos \theta) r \ dz \ dr \ d\theta \\
= \frac{1}{40\pi} \cdot 2 \int_0^{2\pi} \int_0^2 5r^2 \cos \theta + r^3 \cos^2 \theta \ dr \ d\theta \\
= \frac{1}{20\pi} \int_0^{2\pi} \left. \frac{5}{3} r^3 \cos \theta + \frac{1}{4} r^4 \cos^2 \theta \right|_0^2 \ d\theta \\
= \frac{1}{20\pi} \int_0^{2\pi} \left. \frac{40}{3} \cos \theta + 4 \cos^2 \theta \right|_0^2 \ d\theta \\
= \frac{1}{20\pi} \left. \left( \frac{40}{3} \sin \theta + 2\theta + \cos(2\theta) \right) \right|_0^{2\pi} \\
= \frac{1}{20\pi} \left. \left( 0 + 4\pi + 1 - 0 - 0 - 1 \right) \right|_0^{2\pi} \\
= \frac{1}{5}.
\]
The values of $\bar{y}$ and $\bar{z}$ can also be computed directly as above, replacing the $x = r \cos \theta$ by $y = r \sin \theta$ and $z$ respectively.

VII.

A) (10) State Green’s Theorem.

**Solution:** If $D$ is a region in $\mathbb{R}^2$ bounded by a finite collection of simple closed curves, $\partial D$ is the positively-oriented boundary of $D$, and $\mathbf{F}(x, y) = (M(x, y), N(x, y))$ is a $C^1$ vector field on $D$, then
\[
\int_{\partial D} M \ dx + N \ dy = \int \int_D N_x - M_y \ dA.
\]
B) (10) Let \( \mathbf{F}(x, y) = (x - y^2, x^2 + y) \). Verify that Green’s Theorem holds for the region \( D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\} \).

Solution: Using the standard parametrization \((x, y) = (3 \cos(t), 3 \sin(t))\) of the boundary circle of \( D \),

\[
\oint_{\partial D} M \, dx + N \, dy = \int_0^{2\pi} (3 \cos(t) - 9 \sin^2(t))(-3 \sin(t)) \\
+ (9 \cos^2(t) + 3 \sin(t))(3 \cos(t)) \, dt \\
= 27 \int_0^{2\pi} \sin^3(t) + \cos^3(t) \, dt \\
= 27 \left( -\frac{2}{3} \cos(t) - \frac{1}{3} \sin^2(t) \cos(t) + \frac{2}{3} \sin(t) + \frac{1}{3} \cos^2(t) \sin(t) \right) \bigg|_0^{2\pi} \\
= 0.
\]

The double integral over \( D \) is

\[
\int \int_D N_x - M_y \, dA = \int \int_D 2x + 2y \, dA.
\]

This can be evaluated in a number of ways. Switching to polar coordinates, for instance,

\[
\int_0^{2\pi} \int_0^3 2r^2 (\cos \theta + \sin \theta) \, dr \, d\theta = 0
\]

since both \( \int_0^{2\pi} \cos \theta \, d\theta = 0 \) and \( \int_0^{2\pi} \sin \theta \, d\theta = 0 \).

C) (5) We can think of a vector field on \( \mathbb{R}^2 \) as a vector field on \( \mathbb{R}^3 \) by making the last component equal to zero at all points. Compute \( \text{curl}(\mathbf{F}) \) for \( \mathbf{F} = (M(x, y), N(x, y), 0) \) and relate your result to the statement of Green’s Theorem.

Solution: We have

\[
\text{curl}(\mathbf{F}) = \nabla \times (M(x, y), N(x, y), 0)
\]

Since \( M, N \) do not depend on \( z \) and because of the zero in the last component,

\[
\text{curl}(\mathbf{F}) = (0, 0, N_x(x, y) - M_y(x, y))
\]

The integrand in the double integral in Green’s theorem is just the last component here.

Comment: In fact, Green’s theorem is a special case of a more general result called Stokes’ theorem in which the plane region is replaced by a surface \( S \) in \( \mathbb{R}^3 \). Stokes’ theorem then asserts the equality of the line integral of \( \mathbf{F} \) over the boundary of \( S \) and the surface integral of \( \text{curl}(\mathbf{F}) \) over \( S \). The surface integral involves integrating \( \text{curl}(\mathbf{F}) \cdot \mathbf{N} \) where \( \mathbf{N} \) is a normal vector to the surface. The normal vector for \( D \) in
\( \mathbb{R}^2 \) is just \( \mathbf{k} = (0,0,1) \) at each point, and \( \text{curl}(\mathbf{F}) \cdot \mathbf{k} \) is exactly the integrand of the double integral in Green’s theorem.

VIII. A function \( f(x, y) \) is said to be harmonic on an open set \( U \) in \( \mathbb{R}^2 \) if it satisfies the equation
\[
f_{xx} + f_{yy} = 0
\]
at all points in \( U \).

A) (5) How does a nondegenerate critical point of a harmonic function fit into our classification? Is it a local maximum, local minimum, or a saddle point? Explain how you can tell from the Hessian criterion.

\textbf{Solution:} Every nondegenerate critical point of a harmonic function is a \textit{saddle point} because the Hessian matrix is
\[
D^2(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & -f_{xx} \end{pmatrix}.
\]
The determinant here is \(- (f_{xx})^2 - (f_{xy})^2 < 0 \).

B) (5) If \( f \) is harmonic, what is true about the line integral of the vector field
\[
\mathbf{F}(x, y) = (-f_y, f_x)
\]
around any simple closed curve in \( U \)?

\textbf{Solution:} Let \( D \) be the region bounded by the simple closed curve. By Green’s theorem, the integral is equal to
\[
\int \int_D (f_x)_x - (-f_y)_y \ dA = \int \int_D f_{xx} + f_{yy} \ dA = 0.
\]

\textbf{Extra Credit (20)}

Suppose you follow a flow line of the vector field \( \nabla f \) for \( f(x, y) \) in the \( xy \)-plane. As you traverse the flow line in the increasing \( t \)-direction, is the corresponding path on the graph \( z = f(x, y) \) going uphill or downhill? Explain. What does the vector field \( \nabla f \) look like near a local maximum of \( f \)? near a local minimum of \( f \)?

\textbf{Solution:} You are always going upward by the most direct route – recall \( \nabla f(a, b) \) gives the direction in which \( f \) is increasing the fastest. The gradient vector field near a local maximum will have all arrows pointing in toward the critical point (flow lines will converge toward the maximum). Near a local minimum, the gradient vector field will be pointing away from the critical point (flow lines will be diverging away from the minimum).

\textit{Have a peaceful and joyous holiday season!}