I. Let $F(x, y) = (x - y^2, xy)$ and $g(t) = (\sin(t), \cos(t) + \sin(t))$.
A) Compute the derivative matrices $D(F)$ and $D(g)$.

**Solution:** By the definition, $D(F)$ is a $2 \times 2$ matrix and $D(g)$ is a $2 \times 1$ matrix:

$$D(F) = \begin{pmatrix}
\frac{\partial}{\partial x}(x - y^2) & \frac{\partial}{\partial y}(x - y^2) \\
\frac{\partial}{\partial x}(xy) & \frac{\partial}{\partial y}(xy)
\end{pmatrix} = \begin{pmatrix} 1 & -2y \\ y & x \end{pmatrix}$$

and

$$D(g) = \begin{pmatrix} \cos(t) \\ -\sin(t) + \cos(t) \end{pmatrix}.$$  

B) Use the Chain Rule to compute $D(F \circ g)(\pi)$.

**Solution:** The Chain Rule says $D(F \circ g)(\pi) = D(F)(g(\pi))D(g)(\pi)$ here. We have $g(\pi) = (0, -1)$ so

$$D(F)(g(\pi)) = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

and

$$D(g)(\pi) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$  

So

$$D(F)(g(\pi))D(g)(\pi) = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$  

C) Compute the composition $F \circ g$ and find its derivative matrix directly to check your answer from part B.

**Solution:** We have

$$(F \circ g)(t) = (\sin(t) - (\cos(t) + \sin(t))^2, \sin(t)(\cos(t) + \sin(t))).$$  

So

$$D(F \circ g)(t) = \begin{pmatrix} \cos(t) - 2(\cos(t) + \sin(t)(-\sin(t) + \cos(t)) \\ \sin(t)(-\sin(t) + \cos(t)) + (\cos(t) + \sin(t)) \cos(t) \end{pmatrix}.$$  

Substituting $t = \pi$, we get:

$$D(F \circ g)(\pi) = \begin{pmatrix} \cos(\pi) - 2(\cos(\pi) + \sin(\pi)(-\sin(\pi) + \cos(\pi)) \\ \sin(\pi)(-\sin(\pi) + \cos(\pi)) + (\cos(\pi) + \sin(\pi)) \cos(\pi) \end{pmatrix}$$

$$= \begin{pmatrix} -1 - 2(-1)(-1) \\ 0 + (-1)(-1) \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 1 \end{pmatrix},$$
since \( \sin(\pi) = 0 \) and \( \cos(\pi) = -1 \).

II. Let \( z = f(x + y, x - y) \) have continuous partial derivatives with respect to \( u = x + y \) and \( v = x - y \). Show that

\[
\frac{\partial z}{\partial x} \frac{\partial z}{\partial y} = \left( \frac{\partial z}{\partial u} \right)^2 - \left( \frac{\partial z}{\partial v} \right)^2.
\]

**Solution:** By the Chain Rule,

\[
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \cdot 1 + \frac{\partial z}{\partial v} \cdot 1,
\]

and

\[
\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} \cdot 1 + \frac{\partial z}{\partial v} \cdot (-1).
\]

Then

\[
\frac{\partial z}{\partial x} \frac{\partial z}{\partial y} = \left( \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \left( \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) = \left( \frac{\partial z}{\partial u} \right)^2 - \left( \frac{\partial z}{\partial v} \right)^2.
\]

This establishes what we wanted to show.

**Comment:** It is also possible to use the Chain Rule a different way to do this problem. If solve the equations \( u = x + y \) and \( v = x - y \) for \( x, y \) in terms of \( u, v \), then we find \( x = \frac{1}{2}(u + v) \) and \( y = \frac{1}{2}(u - v) \). Then by the Chain Rule,

\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{1}{2} + \frac{\partial z}{\partial y} \cdot \frac{1}{2},
\]

and

\[
\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{1}{2} + \frac{\partial z}{\partial y} \cdot \frac{-1}{2}.
\]

Then substituting and expanding the squares

\[
\left( \frac{\partial z}{\partial u} \right)^2 - \left( \frac{\partial z}{\partial v} \right)^2 = \left( \frac{\partial z}{\partial x} \cdot \frac{1}{2} + \frac{\partial z}{\partial y} \cdot \frac{1}{2} \right)^2 - \left( \frac{\partial z}{\partial x} \cdot \frac{1}{2} + \frac{\partial z}{\partial y} \cdot \frac{-1}{2} \right)^2
\]

\[
= \frac{1}{4} \left( \frac{\partial z}{\partial x} \right)^2 + \frac{1}{2} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} + \frac{1}{4} \left( \frac{\partial z}{\partial y} \right)^2
\]

\[
- \left( \frac{1}{4} \left( \frac{\partial z}{\partial x} \right)^2 - \frac{1}{2} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} + \frac{1}{4} \left( \frac{\partial z}{\partial y} \right)^2 \right)
\]

\[
= \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}.
\]

This establishes what we wanted to show again, but note the first way is somewhat simpler.
III. A mountain has the shape of the graph

\[ z = \sqrt{4 - x^2 - \frac{y^2}{4}}, \]

where \( x, y \) are the east-west and north-south map coordinates respectively, and \( z \) is the altitude.

A) If you are standing at the point on the mountain with \((x, y, z) = (1, 2, \sqrt{2})\), which direction should you move (on the map) to increase your altitude the fastest?

Solution: This will be in the direction of the gradient vector of the altitude function at \((x, y) = (1, 2)\). We have

\[
\nabla f(x, y) = \left( \frac{\partial}{\partial x} \sqrt{4 - x^2 - \frac{y^2}{4}}, \frac{\partial}{\partial y} \sqrt{4 - x^2 - \frac{y^2}{4}} \right)
= \left( \frac{-x}{\sqrt{4 - x^2 - \frac{y^2}{4}}}, \frac{-y}{4\sqrt{4 - x^2 - \frac{y^2}{4}}} \right),
\]

so at \((x, y) = (1, 2)\), we have

\[
\nabla f(1, 2) = \left( \frac{-1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}} \right).
\]

The unit vector specifying the direction of the gradient is

\[
u = \left( \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right).
\]

B) At the same point as in part A, in what direction should you move (on the map) to remain at a constant altitude?

Solution: This will be in directions orthogonal to the gradient vector. These are specified by the unit vectors

\[
u = \pm \left( \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right).
\]

(Note: Given \((a, b)\), to get a perpendicular vector \((c, d)\) with \((a, b) \cdot (c, d) = 0\), we can always take \((c, d) = (-b, a)\).)

IV. Let \(x(t) = (t^2/2, t, t)\) and \(y(t) = (t \cos(t), t \sin(t), t)\) be two curves in \(\mathbb{R}^3\).

A) Does the tangent line to \(x(t)\) at \(t = 2\) cross the \(xy\)-plane? If so, where does this occur?

Solution: The tangent line is the line with parametric equations

\[
x(2) + (t - 2)x'(2) = (2, 2, 2) + (t - 2)(2, 1, 1) = (2t - 2, 2, 2).
\]
This crosses the plane $z = 0$ at $t = 0$, and the intersection point is $(-2, 0, 0)$.

B) Show that $y(t)$ (for $t > 0$) is a flow line for the vector field

$$F(x, y, z) = \left(-y + \frac{x}{z}, x + \frac{y}{z}, 1\right).$$

Solution: The curve $\alpha(t)$ is a flow line of $F$ if $\alpha'(t) = F(\alpha(t))$. We have

$$y'(t) = (-t \sin(t) + \cos(t), t \cos(t) + \sin(t), 1).$$

This is the same as what we get by substituting the component functions of the curve into the formula for $F$: $(-t \sin(t) + \cos(t), t \cos(t) + \sin(t), 1)$.

C) Show that the segments of $x(t)$ and $y(t)$ for $a \leq t \leq b$ have the same arclength for all $a, b$.

Solution: We have the arclength for $x(t)$ is

$$\int_a^b \|x'(t)\| \, dt = \int_a^b \sqrt{t^2 + 2} \, dt.$$

Similarly for $y(t)$,

$$\|y'(t)\| = \sqrt{(-t \sin(t) + \cos(t))^2 + (t \cos(t) + \sin(t))^2 + 1}
= \sqrt{(t^2 + 1)(\sin^2(t) + \cos^2(t)) - 2t \sin(t) \cos(t) + 2t \sin(t) \cos(t) + 1}
= \sqrt{t^2 + 2}.$$

Hence the arclength for $y(t)$ is $\int_a^b \sqrt{t^2 + 2} \, dt$ as well.

Extra Credit: The arclength function for $x(t)$ is

$$\|dT/dt\|/\|dx/dt\| = \frac{\sqrt{2}}{(t^2 + 2)^{3/2}}$$

(after simplification!)