

Mathematics 241 – Multivariable Calculus  
Solutions for Exam 2  
October 29, 2007

I. Let  $F(x, y) = (x - y^2, xy)$  and  $g(t) = (\sin(t), \cos(t) + \sin(t))$ .

A) Compute the derivative matrices  $D(F)$  and  $D(g)$ .

*Solution:* By the definition,  $D(F)$  is a  $2 \times 2$  matrix and  $D(g)$  is a  $2 \times 1$  matrix:

$$D(F) = \begin{pmatrix} \frac{\partial}{\partial x}(x - y^2) & \frac{\partial}{\partial y}(x - y^2) \\ \frac{\partial}{\partial x}(xy) & \frac{\partial}{\partial y}(xy) \end{pmatrix} = \begin{pmatrix} 1 & -2y \\ y & x \end{pmatrix}$$

and

$$D(g) = \begin{pmatrix} \cos(t) \\ -\sin(t) + \cos(t) \end{pmatrix}.$$

B) Use the Chain Rule to compute  $D(F \circ g)(\pi)$ .

*Solution:* The Chain Rule says  $D(F \circ g)(\pi) = D(F)(g(\pi))D(g)(\pi)$  here. We have  $g(\pi) = (0, -1)$  so

$$D(F)(g(\pi)) = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

and

$$D(g)(\pi) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

So

$$D(F)(g(\pi))D(g)(\pi) = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

C) Compute the composition  $F \circ g$  and find its derivative matrix directly to check your answer from part B.

*Solution:* We have

$$(F \circ g)(t) = (\sin(t) - (\cos(t) + \sin(t))^2, \sin(t)(\cos(t) + \sin(t))).$$

So

$$D(F \circ g)(t) = \begin{pmatrix} \cos(t) - 2(\cos(t) + \sin(t))(-\sin(t) + \cos(t)) \\ \sin(t)(-\sin(t) + \cos(t)) + (\cos(t) + \sin(t))\cos(t) \end{pmatrix}$$

Substituting  $t = \pi$ , we get:

$$\begin{aligned} D(F \circ g)(\pi) &= \begin{pmatrix} \cos(\pi) - 2(\cos(\pi) + \sin(\pi))(-\sin(\pi) + \cos(\pi)) \\ \sin(\pi)(-\sin(\pi) + \cos(\pi)) + (\cos(\pi) + \sin(\pi))\cos(\pi) \end{pmatrix} \\ &= \begin{pmatrix} -1 - 2(-1)(-1) \\ 0 + (-1)(-1) \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \end{aligned}$$

since  $\sin(\pi) = 0$  and  $\cos(\pi) = -1$ .

II. Let  $z = f(x + y, x - y)$  have continuous partial derivatives with respect to  $u = x + y$  and  $v = x - y$ . Show that

$$\frac{\partial z}{\partial x} \frac{\partial z}{\partial y} = \left( \frac{\partial z}{\partial u} \right)^2 - \left( \frac{\partial z}{\partial v} \right)^2.$$

*Solution:* By the Chain Rule,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \cdot 1 + \frac{\partial z}{\partial v} \cdot 1$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} \cdot 1 + \frac{\partial z}{\partial v} \cdot (-1).$$

Then

$$\frac{\partial z}{\partial x} \frac{\partial z}{\partial y} = \left( \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \left( \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) = \left( \frac{\partial z}{\partial u} \right)^2 - \left( \frac{\partial z}{\partial v} \right)^2.$$

This establishes what we wanted to show.

*Comment:* It is also possible to use the Chain Rule a different way to do this problem. If solve the equations  $u = x + y$  and  $v = x - y$  for  $x, y$  in terms of  $u, v$ , then we find  $x = \frac{1}{2}(u + v)$  and  $y = \frac{1}{2}(u - v)$ . Then by the Chain Rule,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{1}{2} + \frac{\partial z}{\partial y} \cdot \frac{1}{2}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{1}{2} + \frac{\partial z}{\partial y} \cdot \frac{-1}{2}.$$

Then substituting and expanding the squares

$$\begin{aligned} \left( \frac{\partial z}{\partial u} \right)^2 - \left( \frac{\partial z}{\partial v} \right)^2 &= \left( \frac{\partial z}{\partial x} \cdot \frac{1}{2} + \frac{\partial z}{\partial y} \cdot \frac{1}{2} \right)^2 - \left( \frac{\partial z}{\partial x} \cdot \frac{1}{2} + \frac{\partial z}{\partial y} \cdot \frac{-1}{2} \right)^2 \\ &= \frac{1}{4} \left( \frac{\partial z}{\partial x} \right)^2 + \frac{1}{2} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} + \frac{1}{4} \left( \frac{\partial z}{\partial y} \right)^2 \\ &\quad - \left( \frac{1}{4} \left( \frac{\partial z}{\partial x} \right)^2 - \frac{1}{2} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} + \frac{1}{4} \left( \frac{\partial z}{\partial y} \right)^2 \right) \\ &= \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}. \end{aligned}$$

This establishes what we wanted to show again, but note the first way is somewhat simpler.

III. A mountain has the shape of the graph

$$z = \sqrt{4 - x^2 - \frac{y^2}{4}},$$

where  $x, y$  are the east-west and north-south map coordinates respectively, and  $z$  is the altitude.

- A) If you are standing at the point on the mountain with  $(x, y, z) = (1, 2, \sqrt{2})$ , which direction should you move (on the map) to increase your altitude the fastest?

*Solution:* This will be in the direction of the gradient vector of the altitude function at  $(x, y) = (1, 2)$ . We have

$$\begin{aligned} \nabla f(x, y) &= \left( \frac{\partial}{\partial x} \sqrt{4 - x^2 - \frac{y^2}{4}}, \frac{\partial}{\partial y} \sqrt{4 - x^2 - \frac{y^2}{4}} \right) \\ &= \left( \frac{-x}{\sqrt{4 - x^2 - \frac{y^2}{4}}}, \frac{-y}{4\sqrt{4 - x^2 - \frac{y^2}{4}}} \right), \end{aligned}$$

so at  $(x, y) = (1, 2)$ , we have

$$\nabla f(1, 2) = \left( \frac{-1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}} \right).$$

The unit vector specifying the direction of the gradient is

$$u = \left( \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right).$$

- B) At the same point as in part A, in what direction should you move (on the map) to remain at a constant altitude?

*Solution:* This will be in directions orthogonal to the gradient vector. These are specified by the unit vectors

$$u = \pm \left( \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right).$$

(Note: Given  $(a, b)$ , to get a perpendicular vector  $(c, d)$  with  $(a, b) \cdot (c, d) = 0$ , we can always take  $(c, d) = (-b, a)$ .)

IV. Let  $x(t) = (t^2/2, t, t)$  and  $y(t) = (t \cos(t), t \sin(t), t)$  be two curves in  $\mathbf{R}^3$ .

- A) Does the tangent line to  $x(t)$  at  $t = 2$  cross the  $xy$ -plane? If so, where does this occur?

*Solution:* The tangent line is the line with parametric equations

$$x(2) + (t - 2)x'(2) = (2, 2, 2) + (t - 2)(2, 1, 1) = (2t - 2, t, t)$$

This crosses the plane  $z = 0$  at  $t = 0$ , and the intersection point is  $(-2, 0, 0)$ .

B) Show that  $y(t)$  (for  $t > 0$ ) is a flow line for the vector field

$$F(x, y, z) = \left(-y + \frac{x}{z}, x + \frac{y}{z}, 1\right).$$

*Solution:* The curve  $\alpha(t)$  is a flow line of  $F$  if  $\alpha'(t) = F(\alpha(t))$ . We have  $y'(t) = (-t \sin(t) + \cos(t), t \cos(t) + \sin(t), 1)$ . This is the same as what we get by substituting the component functions of the curve into the formula for  $F$ :  $(-t \sin(t) + \cos(t), t \cos(t) + \sin(t), 1)$ .

C) Show that the segments of  $x(t)$  and  $y(t)$  for  $a \leq t \leq b$  have the *same arclength* for all  $a, b$ .

*Solution:* We have the arclength for  $x(t)$  is

$$\int_a^b \|x'(t)\| dt = \int_a^b \sqrt{t^2 + 2} dt.$$

Similarly for  $y(t)$ ,

$$\begin{aligned} \|y'(t)\| &= \sqrt{(-t \sin(t) + \cos(t))^2 + (t \cos(t) + \sin(t))^2 + 1} \\ &= \sqrt{(t^2 + 1)(\sin^2(t) + \cos^2(t)) - 2t \sin(t) \cos(t) + 2t \sin(t) \cos(t) + 1} \\ &= \sqrt{t^2 + 2}. \end{aligned}$$

Hence the arclength for  $y(t)$  is  $\int_a^b \sqrt{t^2 + 2} dt$  as well.

*Extra Credit:* The arclength function for  $x(t)$  is

$$\|dT/dt\|/\|dx/dt\| = \frac{\sqrt{2}}{(t^2 + 2)^{3/2}}$$

(after simplification!)