Plan for today:

Recap of ideas of symmetry, see how they might apply in music.

Examples from J. S. Bach, A Musical Offering

Discussion – was Bach “doing mathematics?” If not, what was he doing?
Mathematical symmetry

- The precise mathematical understanding of symmetry: *invariance under a transformation.*
- For instance, bilateral symmetry is invariance under reflection across a plane in three dimensional space (or across a line in the plane).
- Other types of transformations can be considered and other symmetries studied.
- Geometrical examples: translations, rotations. Each gives a corresponding form of symmetry.
Visual symmetry – M.C. Escher
How does this apply to music?

- The idea is that various “dimensions” of a musical composition
  - the points in time when different events occur
  - duration and spacing of events
  - the pitches that sound
  - dynamics, instrumentation, expression directions, ...

- can also be subjected to various types of transformations such as augmentation or diminution (stretching out or speeding up time), retrograde motion (time reflection), transposition or inversion in pitch, ...
Symmetry in Music

- These transformations can be used by composers to build up pieces locally and even globally.
- A musical composition can even exhibit the kind of invariance (or "idealized" invariance) that we see in patterns such as the Escher drawing.
An extended example

- We have been discussing possible transformations of music and symmetry properties from a more or less “theoretical” point of view so far.
- The next goal is to explore an extended series of examples to see some of these ideas “in action”.
- Perhaps the best single example -- *A Musical Offering*, by Johann Sebastian Bach (1685-1750).
Background on Bach

• Lived 1685 – 1750 C.E.
• Worked at times as a composer/performer of music in courts of small principalities in present-day Germany.
• Most important position – “Cantor” at St. Thomas in Leipzig (a Lutheran church)
• At Leipzig – he was organist, choirmaster, wrote a tremendous amount of music for church services, also teacher in the St. Thomas choir school.
More on Bach

- Best known as a keyboard (organ, harpsichord, etc.) performer and improviser during his lifetime.
- Worked in a style that was becoming *old-fashioned* even during his lifetime.
- Baroque period roughly 1600 – 1750.
- Bach was the “last gasp” and summing up of this tradition.
Important aspects of Bach's music

- *Polyphony* (the interplay of several musical voices sounding together) and *counterpoint* (musical techniques for doing that) key preoccupations.
- Especially toward the end of his life, he tended to concentrate on the “strictest” contrapuntal forms – *canon* and *fugue*
- “*Pure music*” – often limited directions given for how it was to be performed
History of “The Musical Offering”

- May 7, 1747 – J.S. Bach visits the palace of Frederick II of Prussia (“Frederick the Great”) at Potsdam (outside Berlin).
- Frederick had several new pianofortes made by Gottfried Silbermann; recall Bach was known at the time primarily as a performer.
- Several of his sons (including Carl Phillip Emmanuel, who was employed by Frederick) were better known composers.
History, continued

• Frederick was himself an accomplished flutist and composer of flute music (all in very up-to-date “galant,” “rococo” style);
• But apparently he was also a connoisseur of older music and asked Bach to improvise a fugue in 3 voices on a theme he provided.
• After Bach succeeded there, was also asked to do the same in 6 voices(!) He “begged off” on that one!
The “Royal Theme”

- This is (the basic form of) the fugue subject that Frederick posed to Bach during his visit.
- It apparently stimulated Bach's creative “juices” in a major way – he composed a finished version of the improvised 3-voice fugue, a 6-voice fugue, 10 canons, and a standard 4-movement trio sonata (for flute, violin, harpsichord/cello) all making use of this theme, had all this music printed and sent it to Frederick – a "Musikalisches Opfer"
The canon at the unison

What symmetry is involved here?
A two-voice canon
Structure of this canon

- The piece is 18 measures long (played twice on recording).
- In measures 1 – 9, the top voice plays the Royal Theme, while the bottom voice plays an accompanying figure, with faster motion.
- In measures 10 – 18, the top voice switches to the accompanying figure, while the lower voice plays the Royal Theme, *but both are moving backwards in time*.
- *Also called a canon cancrizans = “crab canon”*
In other words, this canon is a musical “palindrome” -- as a whole piece of music, it is symmetric under reflection in time around the barline between measures 9 and 10. It would sound the same played forwards or backward in time(!)

Another point: Who's Kirnberger? We didn't mention this before, but one curious feature of Bach's *Musical Offering* was that the canon sections were actually left as puzzles for other musicians to decipher. Kirnberger was the first solver!
Bach's actual *puzzle* canon notation

- http://mathcs.holycross.edu/~little/Montserrat1011/MusOffCanon1.pdf
The canon by contrary motion

- What symmetry transformation is involved here?
The modulating canon

"As the notes rise, so may the glory of the King" -- was Bach being ironic?
Some food for thought

- Early examples of canons in medieval music definitely have an element of religious symbolism -- the first voice sets out The Law in musical terms and the other voices follow obediently.

- What would you expect music written to flatter a royal patron and extol his “greatness” to sound like? Do these pieces sound like that?
Questions for further thought

- Not all musicians use these ideas, and even those who do don't *always* use them. What might be a reason for incorporating these ideas in a piece?
- When they do, are they “doing mathematics”?
- “Music is the hidden arithmetical exercise of a mind unconscious that it is calculating” – Gottfried Leibniz.
A final thought

Is this sort of “play” with symmetry transformations limited to art, mathematics, and music? Are we perhaps saying something about the way human intelligence works in general?