MONT 108N – Mathematics Through Time
Selected Solutions – Problem Set 4
November 14, 2010

II. C) Method 1: Let $A_1A_2 \cdots A_n$ be a convex polygon with n sides and n vertices, where the vertices A_i are numbered clockwise around the perimeter, and the edges are A_1A_2, $A_2A_3, \ldots, A_{n-1}A_n$ and A_nA_1. Subdivide the polygon into triangles by introducing new straight lines $A_1A_3, A_1A_4, \ldots, A_1A_{n-1}$ using Postulate 1 for each line. There are $n-3$ such lines and the original polygon is subdivided into $n-2$ triangles (since there is one triangle on each side of each of the new lines). We claim that the sum of the angles in the $n-2$ triangles equals the sum of the interior angles in the original polygon. The angles at A_2 and A_n in the first and last triangles equal the corresponding angles in the original polygon. The angle at A_3 in the polygon is $\angle A_2A_3A_4$, which equals the sum

$$\angle A_2A_3A_1 + \angle A_1A_3A_4.$$

Similarly, for each i, $i = 3, \ldots, n-2$,

$$\angle A_{i-1}A_iA_{i+1} = \angle A_{i-1}A_iA_1 + \angle A_1A_iA_{i+1}$$

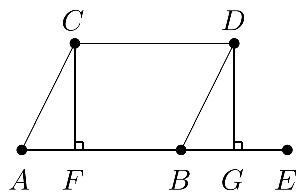
which is the sum of angles in the triangles. Finally, the angle at A_1 in the polygon is

$$\angle A_nA_1A_2 = \angle A_nA_1A_{n-1} + \angle A_{n-1}A_1A_{n-2} + \cdots + \angle A_3A_1A_2.$$

Hence the sum of the angles in the $n-2$ triangles equals the sum of the angles in the original polygon. By part B, this means the sum of the angles in the polygon is $(n-2)$ straight angles.

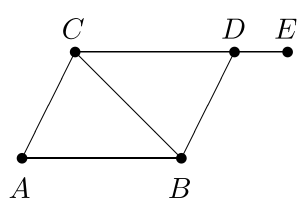
Method 2 (proof by induction): The base case for the induction is the case of a triangle. By part B, the sum of the angles is one straight angle, which is $n-2 = 3-2$ straight angles for a figure with $n = 3$ sides. So the formula works in that case. Now suppose the result has been proved for all convex polygons with k sides and consider a polygon $A_1A_2 \cdots A_{k+1}$ with $k+1$ sides. We number the vertices in the same fashion as in the other proof. If we connect A_1 and A_k with a straight line (Postulate 1), then the $(k+1)$-sided polygon is subdivided into a triangle $\triangle A_1A_kA_{k+1}$ and a k-sided polygon. By induction, the sum of the angles in the triangle and the k-sided polygon is $1 + (k-2) = (k+1) - 2$ straight angles. And as in the first proof, the angles at A_1 and A_k in the two smaller polygons sum to the corresponding angle in the larger polygon. Therefore we have shown that the sum of the angles in any convex n-sided polygon is $n-2$ straight angles.

III. A) Let $ABCD$ be a parallelogram. Using Postulate 2 extend the side AB to a straight line so that applying Proposition 12 to drop perpendiculars from C and D, we obtain points F and G on the straight line containing AB.



Proposition 34 implies that in the parallelogram $ABCD$, $AC = BD$. By the definition of a parallelogram, AC is parallel to BD. Then Proposition 29 implies that $\angle CAF = \angle DBG$. We also have $\angle CFA = \angle DGB$ since both are right angles (Postulate 4). Hence, by Proposition 26 (the AAS congruence criterion), $\triangle ACF$ and $\triangle BDG$ are congruent, and hence have the same area. Moreover, since $AF = BG$ in these congruent triangles, the Common Notions imply that $CD = FG$. Now consider the quadrilateral $FGCD$. This is a rectangle since CF and DG are parallel (Proposition 28), and all the interior angles are right angles (Propsition 29). Moreover $ABCD$ and $FGCD$ have the same area since the parallelogram is $ACF + CFBD$ and the rectangle is $CFBD + BDG$ (Common Notion 2). Therefore the area of the parallelogram is equal to $(FG)(CF) = (AB)(CF)$ which is what we wanted to show.

B) Given a triangle $\triangle ABC$.



Using Proposition 31 (and Postulate 2), construct straight line CE parallel to AB. Then use Proposition 3 to mark off CD equal to AB along line

CE. Then join BD using Postulate 1. Since $CD = AB$ along parallel lines, Proposition 33 implies that $DB = CA$ and they are parallel too. Thus $ABCD$ is a parallelogram by definition. Now since CB is a straight line falling on parallel lines, $\angle ABC = \angle DCB$. Since $AB = CD$ and the side BC is shared, this shows $\triangle ABC$ and $\triangle BCD$ are congruent (Proposition 4). Therefore, the area of $\triangle ABC$ is half the area of the parallelogram $ABCD$, which is AB times the altitude (the perpendicular distance from C to AB as in the previous part).

C. In the figure for part B, suppose the angle at A is actually a right angle. Then we are in the special case in which the altitude equals the side AC. Then part B implies the area is $\frac{1}{2}(AB)(AC)$.