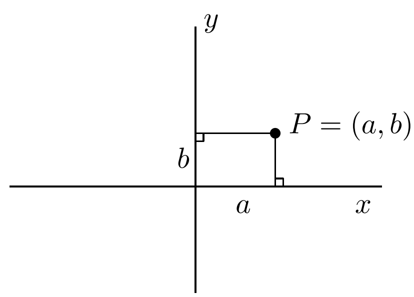
MONT 108N – Mathematics Through Time
Discussion 4/Problem Set 5 – Coordinate Geometry
November 22, 2010

Background

Euclidean-style geometry, where there is no notion of numerical values for lengths or areas is often called *synthetic* geometry. The French mathematician and philosopher *René Descartes* (1596 - 1650 C.E.) thought that the Euclidean formulation made it too difficult to prove results about geometric figures (at least once they got more complicated than straight lines, polygons, and circles). His solution was to introduce numerical *coordinates* for the points in the plane, which were horizontal and vertical distances measured with respect to a pair of *coordinate axes*. The resulting system of xy-coordinates is often called a *Cartesian* coordinate system in his honor. The idea is contained in the following picture:



This idea has become so standard by now that even the typesetting software I am using to create this document uses (a version of) Descartes' idea to create pictures like the one above(!) For instance, the commands that produced the picture above look like this in this system (called LaTeX):

```
\begin{picture}(100,120)
\put(60,10){\line(0,1){100}}
\put(-10,50){\line(1,0){140}}
\put(120,40){$x$}
\put(63,110){$y$}
\put(90,70){\circle*{4}}
\put(95,70){$P = (a,b)$}
\put(75,40){$a$}
\put(53,57){$b$}
\put(90,70){\line(0,-1){20}}
\put(90,70){\line(-1,0){30}}
\put(90,53){\line(1,0){3}}
\put(93,53){\line(0,-1){3}}
\put(63,70){\line(0,-1){3}}
\put(63,67){\line(-1,0){3}}
\end{picture}
```

Can you guess what each command does?

Any such way of doing geometry by means of coordinates is known as as sort of *analytic geometry*. (There are other ways of giving coordinates as well, including the *polar* (r, θ)-coordinates that some of you may know.)

Coordinate Geometry – Discussion Questions

One of the features of Descartes' system is that geometric figures may now be defined as the sets of solutions of *algebraic equations*. This means that Cartesian coordinates let us do *algebraic geometry* (as opposed to the Babylonians' supposed *geometric algebra*).

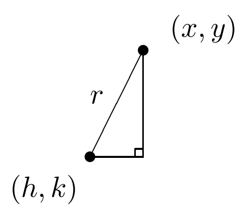
A. *Straight Lines*

1. What sort of xy-coordinate equation is satisfied by the points of a *straight line* in the plane? Are there special cases not covered by the general form(s)? What do the various constants in your equations tell you about the line in geometric terms?

2. If A, B, C are constants and not both A, B are zero, then say why the equation $Ax+By+C = 0$ on the coordinates of a general point (x, y) is satisfied if and only if (x, y) is on a straight line. How does this form of equation relate to what you said in part 1?

B. *Circles* – Following Euclid's Postulate 3, suppose we construct a circle with center at a general point O with coordinates (h, k), and the radius has length $r > 0$.

1. What xy-equation satisfied by all the points (x, y) on the circle? Hint: Consider the following figure



constructed with a line through (h, k) parallel to the x-axis and a line through (x, y) parallel to the y-axis (using Euclid's Proposition 31). What are the sides in this right triangle? What relation must hold?

2. Consider any equation of the form $x^2 + y^2 + Ax + By + C = 0$. Show that the set of all solutions is either a circle, a point, or empty, depending on the relative sizes of the constant coefficients A, B, C. Hint: complete the square in x and y.

C. *Parabolas* – A parabola can be defined in geometric terms as the *locus* (that is, the whole collection) of points that are equidistant from a given point (the focus), and a given line (the *directrix*).

1. For instance, let the focus be the point with coordinates $(0, p)$ on the y-axis with $p > 0$, and let the directrix be the line with equation $y = -p$. The distance from a point $P = (x, y)$ to this line is just the distance along the perpendicular from P to the directrix. Draw a picture showing the points that satisfy the condition defining the parabola.

2. Find the coordinate equation of the parabola.

D. *Ellipses* – This one is harder, but it illustrates the true power of the Cartesian coordinate system. Ancient Greek mathematicians (especially Apollonius) were able to handle conic sections like this by Euclidean methods, but their approach was among the most complicated and advanced mathematics of the time. One way to define an ellipse is this: An ellipse is the locus of all points (x, y) such that *the sum of the distances from (x, y) to two fixed points (called the foci) is a constant.*

1. Suppose the foci are at the points $(-1, 0)$ and $(1, 0)$ and the sum of the distance from (x, y) to $(-1, 0)$ and the distance from (x, y) to $(1, 0)$ is 4. Work out an xy-equation for the resulting ellipse that *has no square roots*. (This will require squaring *twice*, and carefully collecting and cancelling terms at the end.)

2. You might recall from high school geometry that the general equation for an ellipse with foci at points $(\pm c, 0)$ on the x-axis can be put into the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

   for some constants $a > b$ (the semi-major and semi-minor axes). What are those constants for the ellipse you found in part 1, and what do they tell us geometrically about the ellipse?

Assignment

You will discuss these questions in the discussion groups in class on Monday, November 22. But these problems will be due *as an individual problem set* on Friday, December 3.