MONT 108N – Mathematics Through Time
Discussion 1 – Mathematical Discovery
September 17, 2010

Background

There is no doubt that mathematics requires careful, precise, logical thinking (at least if you want to produce reliable, correct results!) But is that (i.e. logical thinking) the *whole story* as Kline sometimes seems to imply? Can there also be *creativity* or the *excitement of finding out something new* (or at least new to you) in doing mathematics? We'll try to see today by looking at a couple of questions – one from geometry, and one from arithmetic. Don't be afraid to get interested/excited if you feel like it! Note: Each group will be assigned *one* of the following questions and work on it for the full period. We will discuss the results on Monday.

Questions

I. Geometric figures with four straight line segment edges ("quadrilaterals") in the plane can have pretty "random" shapes.

1) On the graph paper supplied, draw the quadrilaterals with corners ("vertices") at the points given.
   (a) $(0,0),(1,0),(2,3),(1,5)$ (connect the points with straight line segments in the order given and then connect $(1,5)$ back to $(0,0)$ to close up the figure).
   (b) Do the same for $(0,0),(1,0),(5,1),(3,6)$.
   (c) Do the same for $(0,0),(2,-1),(3,0),(0,5)$.

2) For each of the quadrilaterals from question 1, figure out where the *midpoints* of the 4 edges are, and plot them in your figures. Make a new quadrilateral in each case by connecting the midpoints in order, going around the figure in the same counter-clockwise way as in the original quadrilateral.

3) Do you notice something (surprising) about the quadrilaterals with corners at the midpoints? Are their shapes as "random-looking" as those of the original quadrilaterals? Are these all some particular type of quadrilateral? How can you tell?

4) (Harder) Can you prove a *general statement* about what happens with the process described in part 2)? Suggestion: Try using coordinates with the first two vertices at $(0,0),(1,0)$ (In case you are worried, there is no loss of generality in doing this since we can shift and rotate any quadrilateral to put one side along the positive x-axis starting at $(0,0)$, and then we can choose our unit of length to put the other endpoint of that first edge at $(1,0)$). Then call the third and fourth vertices $(a,b),(c,d)$ in general. Where are the midpoints? What are the edges of the midpoint quadrilateral?

5) All of the figures in part 1) are *convex* quadrilaterals (this means that none of the edges cross, and none of the vertices is inside the triangle formed by the other three vertices). But you can certainly get nonconvex quadrilaterals with the coordinates from part 4). For instance, take $(a, b) = (0, 3)$, $(c, d) = (1, 1)$. Is the general statement still true in this case? What about $(a, b) = (0, 3)$, $(c, d) = (1, 3)$ where edges of the original quadrilateral will cross when you connect the vertices?

II. In class earlier this week we talked about base 2 (binary) and base 8 (octal) as ways to represent whole numbers (and also fractions, etc.) These seem quite different – are they related somehow?

1) Find the base 2 and base 8 forms for the numbers 89, 497, and 1045. Any obvious pattern? (Your answer should be no!)

If you take the base 2 form of a number, you can split it up into 3-digit blocks starting from the 1's digit. For instance 1101101010 $\leftrightarrow$ 001 101 101 010 – note that I inserted two zeros at the start of the leftmost block so that there would always be exactly three binary digits in each block. Now, a three-digit binary number is the same as a base 10 number 0, 1, 2, 3, 4, 5, 6, or 7. These are exactly the same as the base 8 digits! So we can "read a base 2 number as a base 8 number" if we convert each 3-binary digit block to the corresponding base 8 digit. For instance:

$$001\ 101\ 101\ 010 \mapsto 1\ 5\ 5\ 2$$

2) What base 10 number is $(11011010101)_2$. What base 8 number do we get from the 3-digit blocks above – $(1552)_8$? Are these two numbers the same?

3) What happens if you compare the base 2 and base 8 forms of the three numbers from part 1)? Is it the same pattern?

4) (Harder) Can you prove a *general statement* about the relation between the base 2 and base 8 forms of a whole number along these lines? Start by trying to show exactly what happens if there are two blocks of three binary digits(!)

5) Another common number system in the computer world is base 16 ("hexadecimal"). The "digits" in base 16 are written as

$$0, 1, 2, \ldots, 9, 10 = A, 11 = B, 12 = C, 13 = D, 14 = E, 15 = F.$$

If you have a number written in base 2, how can you get its base 16 form (without too much calculation!)?