“Mathematical Thinking” in Art?

- Since start of semester, we have looked at a number of examples from various cultures that (at least possibly) suggest “mathematical thinking” at work:
  - *Calendars* – linear and cyclical time
  - *Maps and Models* – “analogical space”
  - *Games* – recreational mathematics
- Today, we want to continue, by considering several sorts of examples from *visual art*.

Some history

- After the fall of the Western Roman Empire, the Iberian peninsula (today's Spain and Portugal) were parts of various Visigoth kingdoms
- 622 C.E. – the Prophet Muhammad flees from Medina to Mecca (start of Islamic era)
- 637 – 670 C.E. under first Muslim caliphs, Syria, Egypt, Persia, Libya, Morocco were quickly conquered by Arab armies
- 711 C.E. Iberian peninsula captured as well

Al Andalus – Islamic Spain

- 756 – 1031 C.E. *Umayyad caliphate* rules in Spain (capital in Cordoba – the famous Great Mosque still exists there); various successors.
- *Al Andalus* never quite covered the entire peninsula, though, and Christian rulers of states in the north started to attempt to reconquer territories almost immediately (the "Reconquista").
- Another long and complicated history.

Last stages

- From the mid 13th century, the remaining Islamic state in Spain was in the south – the emirate of Granada (actually a vassal state of Christian kingdom of Castile).
- 1492 – Granada conquered by Ferdinand of Aragon and Isabella of Castile and Muslims expelled (most went to North Africa).

Culture of *Al Andalus*

- The Muslim rulers of *Al Andalus* generally encouraged the development of science, philosophy, and art to a very high level.
- Just one example: *Ibn Rushd* (1126 – 1198 C.E.) – known in the rest of Europe by the Latinized name *Averroës* – a very famous philosopher of this period
- Essentially reintroduced *Aristotle's* writings to Western Europe (and sought to reconcile them with a religious point of view).

Patrons of the arts, too

- Averroës wrote extensive commentaries on Aristotle, including especially work on natural world, physics, sciences.
- Was one of the more modern philosophers pictured in Raphael's *School of Athens* that we discussed last semester(!)
- The rulers of *Al Andalus,* especially the emirate of Granada, were also great builders and patrons of the arts
- Their main palace – the *Alhambra*

The *Alhambra*

- (from Arabic – “the red one”) – 14th century

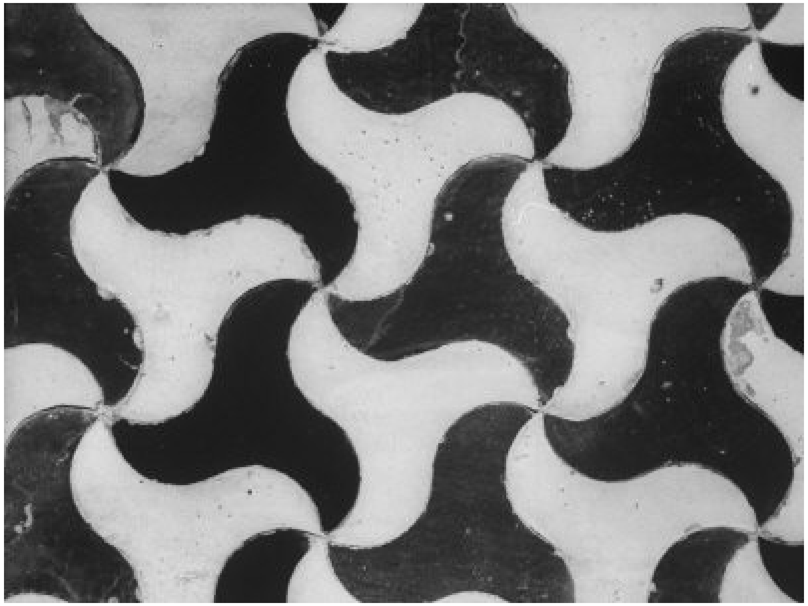
Tile decorations

- Fantastic tile work wall decorations cover almost all of the interior of the *Alhambra*
- The Qur'an (like the Old Testament) prohibits *idolatry* – Islamic artists were not permitted to attempt to *represent God,* or the Prophet (or later -- and by extension – other humans or animals)
- Encouraged the development of very elaborate *abstract*, *geometrical* forms
- (also *calligraphic* forms based on Arabic script – religious symbolism tied to the Qur'an)

An *Alhambra* tile design

Another *Alhambra* design

Yet Another



Tile work, in context – *arabesque*

Influences

- The *Alhambra* has been well-known and treasured since it was created
- Now a UNESCO World Heritage site, a Spanish heritage site, etc.
- Visited by many European and other artists over the years
- In particular, a “life changing event” for the Dutch artist *Maurits Cornelis Escher*, when he visited in first in 1922, and again in 1936.

Escher

- Absolutely fascinated by the different geometric patterns embodied in the *Alhambra* mosaics
- Made a years-long study of *all the ways* such patterns might be constructed
- Incorporated them in many of his own drawings and prints (together with fanciful animals, plants, etc. – he was not bound by the same religious restrictions on figurative elements in his art)

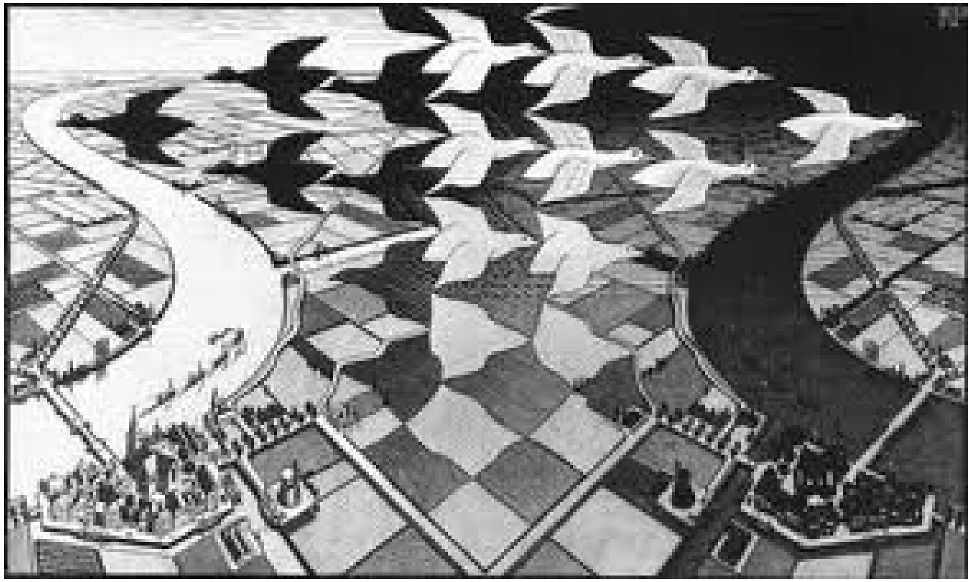
A typical Escher drawing

Escher – fish and birds

Escher – sea shells and starfish

Escher – *Day and Night*

- Escher also used similar patterns in more complex ways:



Repeating patterns

- Almost the defining property of the *Alhambra* tile patterns is that they can be *continued* to cover up as much wall surface as desired
- Notice that the Escher drawings and prints have exactly the same property (except for *Night and Day,* of course).
- A *basic unit* is shifted repeatedly to form the whole pattern in each case.
- Says – *if extended indefinitely,* the whole pattern would be preserved by such a shift.

A mathematical idea

- *Symmetry* is the property of *invariance under a transformation*
- *Example* – bilateral symmetry (like the approximate symmetry of our bodies) is invariance under a *reflection* (mirror image)
- Let's go back and look at some of the Alhambra mosaics and Escher drawings/prints from this point of view.

Observations

- In addition to shifts (or *translations*), some patterns have *additional symmetries*
  - The black and white tile pattern has 120-degree *rotational* symmetry around centers where three of the black or white tiles meet
  - The Escher sea shell and starfish drawing has 90-degree and 180-degree *rotational* symmetries
  - Some patterns (like first one from *Alhambra*) have *reflection* symmetry

Mathematics from symmetry

- Starting in the early 19th century, mathematicians have studied symmetry
- Key tool: the *algebraic properties* of the collection of *all symmetries* a pattern or object has.
- Denote a pattern by *X* (think of it as a set of points in the plane, for instance)
- Consider *distance-preserving mappings* of the plane (*called* *isometries*) – translations, rotations, reflections are all examples.

Mathematics from symmetry, cont.

- Given *X* in the plane, let's denote by *Symm(X)* the collection of all isometries *S* such that *S(X) = X.*
- Example: Let *X* be collect of all points in the square with corners at (1,1), (-1,1), (-1,-1), (1,-1) in the coordinate plane.
- Then *Symm(X)* consists of:
  - <u>rotations</u> about (0,0) by 0, 90, 180, 270 degrees, together with
  - <u>reflections</u> across *x*- and *y-axes,* and lines $y = x$, $y = -x$.

Properties of *Symm(X)*

- If S, T are symmetries of X, then $S(X) = X$ *and* $T(X) = X$.
- *This implies* $S(T(X)) = S(X) = X$ *as well.* (In other words, the *composition* $S \circ T$ of S and T is *also* a symmetry of X.
- The “identity transformation” that maps every point to itself is a symmetry of X.
- Moreover, $I(S(X)) = S(X) = S(I(X))$ – so the identity transformation is like 0 for addition or 1 for multiplication – an *identity element*

More properties of *Symm(X)*

- Composition of transformations is *associative:* $R \circ (S \circ T) = (R \circ S) \circ T$.
- In other words, for every point *x*, $(R \circ (S \circ T))(x) = R(S(T(x))) = ((R \circ S) \circ T)(x)$.
- *Finally,* for every S in *Symm(X)* there is some T in *Symm(X)* such that $S \circ T = I = T \circ S$.
- That is, T *"undoes"* what S *"does"*
- T is called the *inverse transformation of S.*

In our example

- For instance in the example we saw before, where X was the square in the plane with corners at (1,1), (-1,1), (-1,-1), (1,-1)
- If S = 90-degree rotation, then the inverse of S *is the* 270-degree rotation.
- If S = x-axis reflection, then the inverse of S is S *itself.*
- (This is possible – it just says the composition $S \circ S = I$)

Abstracting from this

- Nowadays, this whole set-up is described by the algebraic concept of a *group.*
- ***Definition:*** A group is any set G together with an operation * that combines pairs of elements of G for which the following hold:
  - *for all x, y in G, x*y is in G*
  - ** is associative: (x*y)*z = x*(y*z) for all x,y,z in G*
  - *G contains an identity element e for * satisfying x*e = e*x = x for all x in G*
  - *Each x in G has an inverse y in G satisfying x*y= y*x = e.*

The key example for us

- Note that everything we said above shows: if X is a set in the plane, then $G = Symm(X)$, together with the operation $* = \circ$ *(composition of transformations)* is a group.
- When the collection of elements of a group G *is finite,* then we can describe the operation $*$ by giving an *operation table.*
- For example, let X be the square from before

A group operation table

- *Symm(X) = {R_0 = I (orange), R_1 (light green), R_2 (darker green), R_3 (cyan), T_1 (y=0: dark blue), T_2 (y =x: violet), T_3 (x =0: pink), T_4 (y = -x: red)}*

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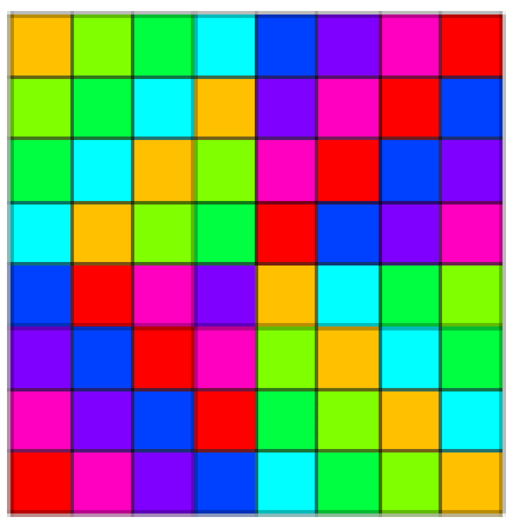

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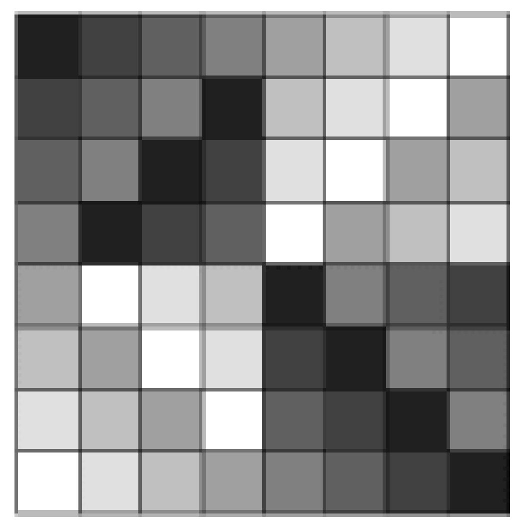

For example

- In coordinates, the 90-degree rotation R_1 is $R_1(x,y) = (y,-x)$.
- The x-axis reflection is $T_1(x,y) = (x,-y)$.
- *So,* the composition $(R_1 \circ T_1)(x,y) = R_1(x,-y) = (-y,-x)$.
- *Thus* $S = R_1 \circ T_1$ *is the reflection across the line* $y = -x$.
- (It satisfies $S \circ S = I$, *and*

$S(1,-1) = (1,-1)$ and $S(-1,1) = (-1,1)$.)

On the other hand

- $(T_1 \circ R_1)(x,y) = T_1(y,-x) = (y,x)$.
- *Thus* $U = T_1 \circ R_1$ *is the reflection across the line* $y = x$.
- (It satisfies $U \circ U = I$, *and*

$U(1,1) = (1,1)$ and $U(-1,-1) = (-1,-1)$.)

- *Note that this group operation is* ***not commutative****:* $T_1 \circ R_1$ *is not the same transformation as* $R_1 \circ T_1$.

“Wallpaper patterns”

- Let's concentrate now on repeating patterns that can be used to “fill out” the whole plane if extended indefinitely
- Called “wallpaper patterns” or *regular tesselations*
- If X is such a pattern, then the symmetry group $Symm(X)$ contains translations in two independent directions, plus possibly other transformations (rotations, reflections across lines)

Groups and classification

- We can use the groups of symmetries *Symm(X)* and *Symm(X')* to give an idea of when two patterns are *formed in the same way* or are “equivalent” in a sense.
- This will be true if there is a one-to-one correspondence between the groups that takes compositions in the first group to the corresponding compositions in the second.
- Leads to a *classification* of all possible repeating patterns by their symmetry groups.

There are exactly 17(!)

- See the link from our course homepage to David Joyce's “wallpaper groups” page.
- Apparently the Islamic artists who created the Alhambra tile work knew about most (or all?) of these – depending on how you extend patterns there, you can find examples similar to all these types.
- Escher essentially recreated this sort of classification too, independently (organized rather differently, though)