MATH 135 – Calculus 1
Indeterminate Forms and Algebraic Evaluation of Limits
September 30, 2019

Background

Most of the limits we will need to deal with in calculus are so-called *indeterminate forms* like $0/0$, ∞/∞ (and eventually also $0 \cdot \infty$ and $\infty - \infty$), where the Limit Laws and continuity don't help immediately. For instance

$$\lim_{x \to 1} \frac{x^4 - x^3}{x^2 - 1}$$

is a $0/0$ form since the numerator and denominator of this rational function both tend to 0 as $x \to 1$. The perhaps surprising thing is that in cases like this we can often

(1) algebraically cancel a factor producing the 0's in the numerator and denominator, and/or

(2) rearrange the function in other ways

to see that a limit exists and evaluate it. For example, the limit above can be evaluated like this. After factoring and cancelling a common factor of $x - 1$, we get:

$$\frac{x^4 - x^3}{x^2 - 1} = \frac{x^3(x-1)}{(x+1)(x-1)} = \frac{x^3}{x+1}.$$

This new function is equal to the original one for all $x \neq 1$, but it is continuous at $x = 1$. Hence we can see

$$\lim_{x \to 1} \frac{x^4 - x^3}{x^2 - 1} = \lim_{x \to 1} \frac{x^3}{x+1} = \frac{1}{2}.$$

This method will always involve replacing the original function by some other function that is defined and continuous at the x-value where we are taking the limit, and then the limit can be obtained by substitution, *after the replacement.*

The following questions will "lead you through" some first examples. *Eventually, though, you will need to be able to recognize which algebraic methods apply and work through examples like this without the prompts from me.*

Questions

(1) Consider the limit

$$\lim_{x \to 4} \frac{x^3 - 64}{x^2 - 16},$$

(a) First explain why this is a $0/0$ form. $\lim_{x \to 4} x^3 - 64 = 0 = \lim_{x \to 4} x^2 - 16$

(b) Next, factor the top and bottom. You should find a common factor of $x - 4$ that can be cancelled.

$$\frac{x^3 - 64}{x^2 - 16} = \frac{\cancel{(x-4)}(x^2 + 4x + 16)}{\cancel{(x-4)}(x+4)}$$

(c) Do this, and then note that "what is left" is continuous at $x = 4$, so we can find the limit by substitution in the cancelled form. What is the limit? $\lim_{x \to 4} \frac{x^2+4x+16}{x+4} = \frac{48}{8} = \boxed{6}$

(2) Next, consider this limit:

$$\lim_{x\to 3} \frac{\sqrt{x^2+16}-5}{x-3}.$$

(a) Explain why this is also a 0/0 form. $\lim_{x\to 3} \sqrt{x^2+16} - 5 = 0 = \lim_{x \to 3} x-3$

(b) We can look at the top as $\sqrt{x^2+16} - \sqrt{25}$. In cases like this, multiply top and bottom by the conjugate radical $\sqrt{x^2+16} + \sqrt{25}$ and simplify the top. $\frac{(\sqrt{x^2+16}-5)}{(x-3)} \cdot \frac{(\sqrt{x^2+16}+5)}{(\sqrt{x^2+16}+5)} = \frac{x^2-9}{(x-3)(\sqrt{x^2+16}+5)}$

(c) You should now see something to cancel so that the remaining function is continuous at $x = 3$. Do that and determine the limit. $= \frac{x+3}{(\sqrt{x^2+16}+5)}$

$\lim_{x\to 3} \frac{x+3}{\sqrt{x^2+16}+5} = \frac{6}{10} = \boxed{\frac{3}{5}}$

(3) Next consider this limit:

$$\lim_{x\to 5} \frac{\frac{1}{x^2} - \frac{1}{25}}{x-5}$$

(a) Why is this one an indeterminate 0/0 form? $\lim_{x\to 5} \frac{1}{x^2} - \frac{1}{25} = 0 = \lim_{x\to 5} x-5$

(b) Put the two terms on the top over a common denominator. Then simplify the fraction and find something to cancel. $\frac{25-x^2}{25x^2(x-5)} = \frac{-(x-5)(x+5)}{25x^2(x-5)} \Rightarrow \lim_{x\to 5} \frac{-(x+5)}{25x^2} = \frac{-10}{625}$

(c) Evaluate the limit. $= \boxed{\frac{-2}{125}}$

(4) Many ∞/∞ indeterminate form limits can be evaluated in a similar way. We'll consider

$$\lim_{x\to\pi/2} \frac{\tan(x)}{\sec(x)}$$

(a) Explain why this one is an ∞/∞ indeterminate form.

(b) Rewrite in terms of $\sin(x)$ and $\cos(x)$ and simplify, cancelling something in the process.

(c) What is left should be continuous at $x = \pi/2$, so evaluate the limit by substitution.

$\lim_{x\to\frac{\pi}{2}} \tan x = \pm\infty$ and $\lim_{x\to\frac{\pi}{2}} \sec x = \pm\infty$ (the one-sided limits have different signs

$$\frac{\tan(x)}{\sec(x)} = \frac{\frac{\sin(x)}{\cos(x)}}{\frac{1}{\cos(x)}} = \sin(x)$$

so $\lim_{x\to\frac{\pi}{2}} \sin(x) = \boxed{1}$