MONT 104N – Modeling The Environment
Solutions for Final Examination Review Problems – December 6, 2019

I. In 1990, forests covered 4.047×10^9 hectares of the Earth's surface. By 2000, forest area had decreased to 4.038×10^9 hectares. Assuming that the decrease in forest area is linear, and that it will continue at the same rate into the future, in this problem you will develop a linear model for the forest area remaining as a function of $t =$ years since 1990.

A. Determine the slope for the linear model of the forest area.

Solution: The slope is

$$m = \frac{4.038 - 4.047}{2000 - 1990} = -.0009.$$

The slope is in units of 10^9 hectares per year. (This is equivalent to a net loss of 900,000 hectares, or 9000 square kilometers per year.)

B. What is the linear equation modeling the forest area as a function of $t =$ years since 1990.

Solution: Writing FA for the forest area, the model would be

$$FA = 4.047 - .0009t$$

C. (10) According to your model, in what year will the forest area reach 4.0×10^9 hectares?

Solution: We solve for t from the equation:

$$4.0 = 4.047 - .0009t,$$

so

$$t = \frac{4.047 - 4.0}{.0009} \doteq 52$$

This corresponds to 52 years after 1990, so the year 2042.

D. (5) According to the United Nations Food and Agriculture Organization, the actual forest area remaining in 2010 was 4.033×10^9 hectares. How close is the prediction your model from part B gives for the forest area in 2010?

Solution: Using the model equation from part B to predict the amount of forest area that will remain in 2010: 2010 is $t = 20$ years after 1990, so the model predicts

$$FA = 4.047 - (.0009)(20) = 4.029 \; (\times 10^9 \text{ hectares})$$

The rate of deforestation was slower between 2000 and 2010 than the rate between 1990 and 2010. So the model predicted a lower forest area remaining than the actual figure.

II. Wind power has emerged as a fast growing source of energy for electrical power generation in recent years. In 2016, the generating power of wind turbines installed around the world was about 301 gigawatts and it was increasing at about 33.2% per year.

A. The typical English unit of power is the horsepower. 1 horsepower $= 7.457 \times 10^{-7}$ gigawatts. Convert 301 megawatts to the equivalent number of horsepower.

Solution: Thinking about the unit conversion, we see

$$\text{horsepower} = \text{megawatt} \times \frac{\text{horsepower}}{\text{megawatt}}$$

That is, there are

$$\frac{1}{7.457 \times 10^{-7}} \doteq 1.341 \times 10^{6}$$

horsepower in one gigawatt. Hence 301 gigawatts is

$$301 \times 1.341 \times 10^{6} = 403.641 \times 10^{6} \doteq 4.03 \times 10^{8} \text{ horsepower}$$

B. Construct an exponential model for WP = wind power generation as a function of t = years since 2016. *Use units of* 10^2 *gigawatts for WP* – see the entry for 2016 in the table below.

Solution: In units of 10^2 gigawatts, the model equation is:

$$WP = (3.01)(1.332)^{t}$$

C. Fill in the table of values for WP below with values predicted by your model for the years 2017 – 2021. Round to 2 decimal places. About how many years will it take for WP to reach approximately double the 2016 level?

$Year$	2016	2017	2018	2019	2020	2021
WP	3.01	4.01	5.34	7.11	9.48	12.62

WP will double in between 2 and 3 years. (**Note**: The doubling time for an exponential function with $a = 1.322$ can also be found by the formula in one of the problems sets:

$$1.322 = 2^{\frac{1}{t_2}}$$

where $t_2 = \frac{\log_{10}(2)}{\log_{10}(1.332)} \doteq 2.42$ years.)

D. (5) How many years will it take for wind power generation to reach 20×10^2 gigawatts according to your model?

Solution: The equation we want to solve is $20 = (3.01)(1.332)^t$. This is true when $t = \frac{\log_{10}(20/3.01)}{\log(1.332)} \doteq 6.6$. So the projection is this value will be reached between 2022 and 2023.

III. Suppose that a population of fast-reproducing insects in an area has a natural growth rate of 5% per month from births and deaths, and that there is a net migration *loss* of 20 individuals per month.

A. Write a difference equation that models this situation.

Solution: The equation would be

$$P(n+1) = (1.05) \cdot P(n) - 20$$

B. Using an initial value $P(0) = 500$, determine the populations in months $1, 2, 3, 4, 5$ according to the model you picked in part A and record the values in the following table (round any decimal values to the nearest whole number)

Solution:

n	0	1	2	3	4	5
$P(n)$	500	505	510	516	522	528

C. What happens to the population in the long run as n increase? Does it tend to a definite value?

Solution: It seems that the population will continue increasing at an increasing rate as n increases. If $P(n) > 400$, then $P(n+1) = (1.05)P(n) - 20 > P(n+1)$ and the difference $P(n+1) - P(n) = (.05)P(n) - 20$ is bigger the larger $P(n)$ is. (**Note:** By the formula we discussed in class for the general solution of affine first order difference equations, this difference equation has an equilibrium solution at $P = \frac{20}{.05} = 400$. However, it is an *unstable equilibrium* since $1.05 > 1$. This means that with an initial value $P(0) = 500 > 400$, the solution will grow without bound as n increases. In a question like this, I could also ask you to find the equilibrium level.)

IV. Answer the following questions with a few sentences each.

A. If you are fitting an exponential model to a data set (x_i, y_i) "by hand," you start by transforming the data to $(X, Y) = (x_i, \log_{10}(y_i))$. If the best fit regression line for the transformed data is $Y = mX + b$, what is the corresponding exponential model? (Assume the logarithms have base 10 as we discussed in class.)

Solution: The linear equation is equivalent to $\log_{10}(y) = mx+b$ in terms of the original variables. So exponentiating both sides we get $y = 10^b \cdot (10^m)^x$. In other words, 10^m is the base a of the exponential function, and 10^b is the constant multiplier.

B. If you are fitting a power law model to a data set (x_i, y_i) "by hand," you start by transforming the data to $(X, Y) = (\log_{10}(x_i), \log_{10}(y_i))$. If the best fit regression line for the transformed data is $Y = mX + b$, what is the corresponding power law model? (Assume the logarithms have base 10 as we discussed in class.)

Solution: The linear equation is equivalent to $\log_{10}(y) = m \log_{10}(x) + b$ in terms of the original variables. So exponentiating both sides we get $y = 10^b \cdot x^m$. In other words, m is the exponent, and 10^b is the constant multipler

C. What does the R^2 statistic in measure in linear regression? How did we use it? Explain what it would mean, for instance if $R^2 = 1$.

Solution: The R^2 statistic measures the degree of linearity in a scatter plot (in other words, how close the data points come to lying on a single straight line). If $R^2 = 1$, then all the points are on a single line. We used this to measure the goodness of fit even for exponential models. When we did this, we were looking at the correlation coefficient for the *transformed* data (the $(x_i, \log_{10}(y_i))$ in the exponential case).

D. What type of chart (scatterplot, pie chart, bar chart, etc.) would be most useful to describe the composition of a forest if there 5 different types of trees present in different concentrations per acre? Explain, and illustrate your answer with a chart if a typical acre of forest contains 10 oaks, 12 maples, 5 pines, 2 hemlocks, and 1 chestnut.

Solution: For a chart indicating the composition of a whole made up of several parts, either a pie chart or a bar chart could be used. But a *pie chart* would be a slightly superior choice to show the composition. For a pie chart, we would compute the percentages of the whole represented by each species: $10 + 12 + 5 + 2 + 1 = 30$ trees. So oaks account for $10/30 \times 100\% = 33.3\%$, maples account for $12/30 \times 100\%$, or 40%, pines account for $5/30 \times 100\%$, or 16.7%, hemlocks account for 6.7%, and chestnuts account for the remaining 3.3%. These would be shown as fractions of a whole circle in the pie chart.

E. What difference equation would model a population undergoing logistic growth if the population was growing at about 4% per year when the population is much smaller than the carrying capacity $M = 400$ of the habitat.

Solution: We see $r = .04$, so equation is

$$P(n+1) = (1+r)P(n) - \frac{r}{M}(P(n))^2 = 1.04P(n) - \frac{.04}{400}(P(n))^2.$$

F. What feature of the solutions of the Lotka-Volterra equations is considered a confirmation that this model is capturing an important aspect of real-world predator-prey interactions?

Solution: The fact that the solutions tend to exhibit *cyclical, oscillatory* behavior is a confirmation of this. The equations were originally developed to model the predator-prey behavior of pairs of species (like the Canada lynx and snowshoe hares that we saw in class). Since those populations are observed to oscillate in the wild, the model is (at the least) doing something similar.

Extra Review Problems from Book

Chapter 1.

5. (a) $44,200 \text{ km}^2 = 44,200 \text{ km}^2 \cdot (.621)^2 \text{ mi}^2/\text{km}^2 \doteq 17045 \text{ mi}^2$.

   (b) The thickness in kilometers is .350 km, so the volume in cubic kilometers $= 44,200 \cdot .350 \doteq 15470 \text{km}^3$. Then converting to cubic feet, 1 cubic kilometer is about $(5280/1.61)^3 \doteq (3280)^3$ cubic feet, so 15470 cubic kilometers is $\doteq 3.53 \times 10^{10}$ cubic feet.

11. (a) $\log_{10}(5.34689) \doteq .7281$, $\log_{10}(53.4689) \doteq 1.7281$ and $\log_{10}(534.689) \doteq 2.7281$. What is happening is that when you multiply by a power 10^k, then $\log_{10}(10^k) = k$ is *added* to the value of the logarithm, by part (1) of Proposition 1.8.

   (b) $\log_7(34.333) = \log_{10}(34.333)/\log_{10}(7) \doteq 1.8172$ (see formula (1.2) on page 12.)

   (c) $\ln(100.3) \doteq 4.6082$. (This could also be computed as $\log_{10}(100.3)/\log_{10}(e)$, where e is the base of the natural logarithms.)

13. (a) $20 \cdot \log_{10}(5000/20) \doteq 47.96$ dB.

   (b) If $L = 1$ dB, then $1 = 20 \cdot \log_{10}(p_m/20)$, so $p_m = 20 \cdot 10^{1/20} \doteq 22.44$ micropascals. Similarly if $L = 2$ dB, then $p_m \doteq 25.18$ micropascals, and if $L = 10$ dB, then $p_m \doteq 63.25$ micropascals.

   (c) About $20 \cdot 10^{150/20} \doteq 6.32 \times 10^8$ micropascals. This shows a logarithmic scale in action!

Chapter 2.

2. (a) First student: absolute error $= 3.44 - 3.40 = .04$ meter. Percent relative error $= \frac{3.44-3.40}{3.40} \times 100\% \doteq 1.2\%$. Second student: absolute error $= .44 - .40 = .04$ meter. Percent relative error $= \frac{.44-.40}{.40} \times 100\% = 10\%$.

   (b) The first student was more accurate because the percent relative error was smaller.

   (c) The absolute error tells you how far off the measurement is from the true value. Absolute errors are important one measurement at a time. The percentage relative error expresses the error as a percent of the exact value. So, percentage relative errors are more useful for comparing accuracies of different measurements.

(d) Precision of a set of measurements is not the same as accuracy. Accuracy measures how far the measurements are from an exact value. Precision measures how close the measurements are to each other.

7. (a) Boston: 13321.3 people per square mile $\doteq$ 34530.1 people per square kilometer. Chicago: 11868.1 people per square mile $\doteq$ 30763.3 people per square kilometer. Miami: 11198.7 people per square mile $\doteq$ 29028.1 people per square kilometer. NYC: 27016.3 people per square mile $\doteq$ 70029 people per square kilometer. Philadelphia: 11233.6 people per square mile $\doteq$ 29118.7 people per square kilometer. San Francisco: 17246.4 people per square mile $\doteq$ 44704.4 people per square kilometer.

(b) In 2016, from page 26, we see that the population of New York was estimated at 8.54×10^6. That gives a population density of 28222.1 people per square mile. The percent change in population density from 2010 (reference) to 2016 (comparison) was

$$\frac{28222.1 - 27016.3}{27016.3} \times 100\% \doteq 4.5\%$$

The numbers would come out the same if the densities per square kilometer were used since the conversion factors from square miles to square kilometers in the numerator and denominator would cancel.

Chapter 4.

6. (a) $y - 5.2 = 7 \cdot (x - 3.2)$ or $y = 7x - 17.2$.

(b) $y - 1 = -1/7(x - 0)$ or $y = -x/7 + 1$.

8. $g(x)$ is the linear one because that is the only function for which the slopes between pairs of points are always the same $= -2.25$. The equation is $y - 1.525 = -2.25(x - 1.1)$ or $y = -2.25x + 4$.

9. Kudzu area $= \frac{7.4-0}{2018-1876}(t - 1876)$ (in units of 10^6 acres, t in years).

Chapter 5.

2. All of these are solved by taking logarithms:

(a) $x = \frac{1}{2}\frac{\log_{10}(28.3/4.5)}{\log_{10}(3.4)} \doteq .7513$.

(b) $x = \frac{\log_{10}(3.5)}{\log_{10}(4)-\log_{10}(2)} \doteq 1.8074$.

(c) $x = \frac{\log_{10}((7.9-5.6)/2.8)}{\log_{10}(7.4)} \doteq -.09828$.

4. (a) $f(t) = 8.54 \cdot (1.15)^t$.
   (b) $f(t) = 3.5711 \cdot (1.03)^t$.

6. (a) doubling time $= \frac{\log_{10}(10/5)}{\log_{10}(1.25)} \doteq 3.106$
   (b) doubling time $= \frac{\log_{10}(36.6/18.3)}{\log_{10}(3.4)} \doteq .5664$.

7. As in the special cases in 6 above, we solve for t in the equation

$$2 \cdot Q(0) = Q(0) \cdot a^t$$

to find the doubling time, t_2. Dividing both side by $Q(0)$, then taking logarithms, we get

$$t_2 = \frac{\log_{10}(2)}{\log_{10}(a)}.$$

8. This follows from rules for exponents. From the previous problem, $\log_{10}(a) = \frac{\log_{10}(2)}{t_2}$, so

$$a = 10^{\log_{10}(a)} = 10^{\log_{10}(2) \cdot \frac{1}{t_2}} = 2^{1/t_2}.$$

Then raising both sides to the t power and multiplying by $Q(0)$ we get

$$Q(t) = Q(0) \cdot a^t = Q(0) \cdot \left(2^{1/t_2}\right)^t = Q(0) \cdot 2^{t/t_2}.$$

9. (a) Proceeding as in Example 5.8 in the text, from the half-life of this isotope:

$$Q(28.8) = \frac{1}{2} \cdot Q(0) = Q(0) \cdot a^{28.8}$$

so $\log_{10}(a) = \frac{\log_{10}(.5)}{28.8} \doteq -.0105$ and $a = 10^{-.0105} \doteq .9762$. Then the model is

$$Q(t) = Q(0) \cdot (.9762)^t.$$

   (b) We solve for t:

$$.01 \cdot Q(0) = Q(0) \cdot (.9762)^t,$$

so $t = \frac{\log_{10}(.01)}{\log_{10}(.9762)} \doteq 191.2$ years.

Chapter 7.

1. The percentage change gives $Q(n+1) - Q(n)Q(n) \times 100 = r$, so $Q(n+1) = \left(1 + \frac{r}{100}\right) \cdot Q(n)$. Hence if we start from $t = 0$, we get

$$\begin{aligned}
Q(1) &= \left(1 + \frac{r}{100}\right) Q(0) \\
Q(2) &= \left(1 + \frac{r}{100}\right) Q(1) = \left(1 + \frac{r}{100}\right)^2 Q(0) \\
Q(3) &= \left(1 + \frac{r}{100}\right) Q(2) = \left(1 + \frac{r}{100}\right)^3 Q(0)
\end{aligned}$$

and so forth. The general pattern is

$$Q(n) = \left(1 + \frac{r}{100}\right)^n Q(0).$$

Note that this is an exponential function of n, with $a = \left(1 + \frac{r}{100}\right)$.

4. (a) $Q(n) = 3.4 \cdot (1.8)^n$ by problem 1 above.

   (b) Here we want to use the general solution for affine first order equations from (7.4) on page 129:

$$Q(n) = \left(4.3 + \frac{(-.03)}{(.78-1)}\right)(.78)^n - \frac{(-.03)}{(.78-1)} = 4.4364 \cdot (.78)^n - .1364.$$

8. (a) From the equation, $r = .03$ and $r/M = .006$, so $M = 5$. Since $Q(0) = .8 < M$, the solution will increase in a sort of S-shape and tend toward M as a horizontal asymptote as n increases.

   (b) $r = .34$ and $r/M = .0009$, so $M = 377.8$. Since $Q(0) = 420 > M$, the solution will decrease toward M.

   (c) $r = .86$ and $r/M = .0048$, so $M = 179.2$. Since $Q(0) < M$, this is similar to part (a).

Chapter 8.

3. The equations are

$$\begin{aligned} A(n+1) &= .4A(n) + .25B(n) + .5C(n) \\ B(n+1) &= .4B(n) + .4A(n) \\ C(n+1) &= .33C(n) + .2B(n) \end{aligned}$$