College of the Holy Cross, Fall Semester 2019
MONT 104N – Modeling the Environment
Solutions for the Final Exam, December 20

I. The table below shows estimates collected in the EDGAR database created by the European Commission and the Netherlands Environmental Assessment Agency of the amounts of *carbon dioxide* emissions from burning of fossil fuels in 2015 and 2016 by country. The units are megatonnes = 10^6 metric tonnes. The populations are in units of millions = 10^6 of people:

Country	CO2 2015	CO2 2016	2015 Population
China	10.642	10.433	1367
United States	5.712	5.012	321
India	2.455	2.533	1252
Russia	1.761	1.662	142
Japan	1.253	1.240	127
Germany	.778	.776	81
Whole world	36.061	35.753	7256

A. 1 metric tonne is 1000 kg and 1 kg $\doteq$ 2.205 lb. What was the equivalent amount of carbon dioxide emissions for India in 2016, in units of pounds.

Answer:

$$(2.533 \times 10^6) \text{ tonnes } \cdot (2205) \text{ lb/tonne } \doteq 5.585 \times 10^9 \text{ lb}$$

B. Suppose that the U.S. carbon dioxide emissions were decreasing *exponentially*. What would be the exponential model fitting the two data points you have exactly? Take $t = 0$ to correspond to the year 2015. What would your model predict for U.S. emissions in the year 2020?

Answer: We are looking for a function of the form $USCO2(t) = ca^t$ for which $USCO2(0) = 5.712$ (from the 2015 figure) and $USCO2(1) = 5.012$ (from the 2016 figure). The first equation says $c = 5.712$ and then $5.012 = 5.172 \cdot a^1$ implies $a \doteq .8775$. Hence, the model has the form $USCO2(t) \doteq 5.712(.8775)^t$ (units are megatonnes CO_2). The prediction for 2020 is $USCO2(5) \doteq 2.972$ megatonnes CO_2.

C. Now, let's look at this data from another perspective. Which countries here had *per capita* CO_2 emissions greater than the world per capita emissions in 2015, and which are less than the world average?

Answer: The world per capita figure is

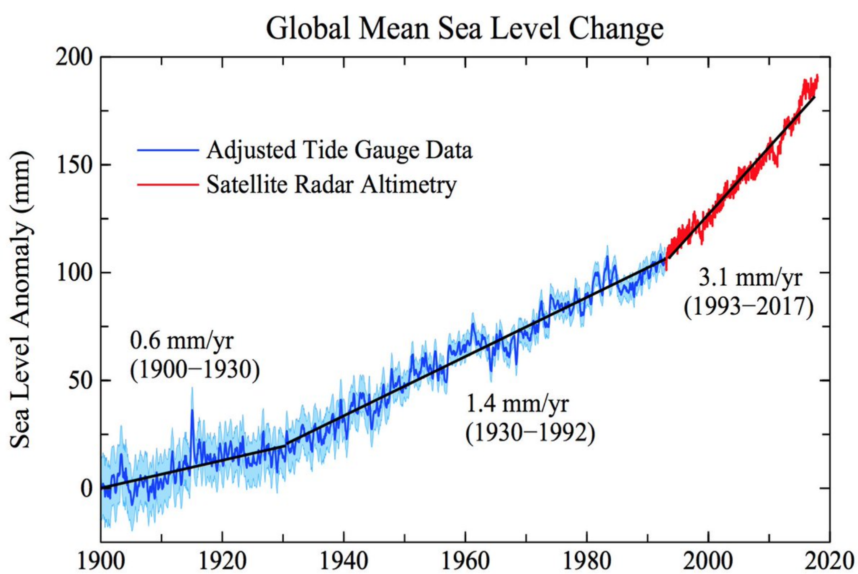
$$\frac{(36.61 \times 10^6) \text{ tonnes}}{(7256 \times 10^6) \text{ people}} \doteq .005 \text{ tonnes per person } \doteq 5 \text{ kg per person }.$$

The country by country averages are as follows (note that I'm not putting in the $\times 10^6$ in the numerators and denominators because they cancel):

Country	**2015 CO2 per capita**
China	$10.642/1367 \doteq .0078$ tonnes per person
United States	$5.712/321 \doteq .018$ tonnes per person
India	$2.455/1252 \doteq .002$ tonnes per person
Russia	$1.761/142 \doteq .012$ tonnes per person
Japan	$1.253/127 \doteq .0098$ tonnes per person
Germany	$.778/81 \doteq .0096$ tonnes per person
Whole world	$36.061/7256 \doteq .005$ tonnes per person

So all of these countries except India have per capita emissions larger than the world per capita figure. Of course, the explanation is that all of these except India are largely industrialized societies, so CO_2 emissions are higher than the world average.

II. The following graph, constructed by Makiko Sato and James Hansen at the Columbia University Climate Science, Awareness and Solutions project, shows the change in *global average sea level* over the period 1900 to 2017 (i.e. the present). It's also a very nice example of how the basic modeling techniques we have discussed can be adapted to deal with dynamic, changing situations.



The vertical axis is the *sea level anomaly*—the *difference* between the observed levels in later years and the average level in 1900.

A. The information included in the graph shows that Sato and Hansen have constructed three separate linear models over three different time periods: one covering the period 1900 - 1930, the second covering the period 1930 - 1992 and the last one covering the period 1993 to 2017. The estimated slopes are given in units of mm/yr. Find the equations of the three linear models (the equations of the three black lines in the graph) for average mean sea level $L(t)$, using $t =$ actual year in each case. Use the values 0 for $L(1900)$, 18 for $L(1930)$, and 104.8 for $L(1993)$.

Answer: The equations are (output in units of mm):

$$L(t) = \begin{cases} .6(t-1900) = .6t - 1140, & 1900 \le t \le 1930 \\ 1.4(t-1930) + 18 = 1.4t - 2684, & 1930 \le t \le 1992 \\ 3.1(t-1993) + 104.8 = 3.1t - 6073.5, & 1992 \le t \le 2017 \end{cases}$$

(Note: the problem said to use $t =$ the actual year in each case, so you need the point-slope form of the equation of the line with ($t -$ starting year) each time. Also, the negative values for the intercepts just mean that if the lines were extended to the left back to the year 0, they would be intersecting the vertical axis way below 0. However, that is essentially a silly thing to do here, since it's extrapolating so far beyond the range of times the data came from.)

B. Why do you suppose Sato and Hansen chose to fit three separate linear models rather than a single exponential or power law model?

Answer: One reasonable explanation is that the rate of increase seems to be quite nearly constant over each of the three time intervals separately. So three linear models are a better description than a single exponential or power law.

C. What is the "take-away" message from this data and this graphic? How would you summarize the conclusion(s) we should draw from it?

Answer: One reasonable take-away message would be that sea-levels have been increasing at a faster and faster rate over the period from 1900 to 2017, and that there have been essentially three different phases or stages in that increase.

III. Suppose that a endangered population of salamanders in a protected wetland is *decreasing* at a net rate of 15% per year from births and deaths, but human-reared salamanders are being reintroduced into the habitat at a rate of 40 individuals per year.

A. Write a difference equation that models the change in the salamander population from each year to the next.

Answer: Letting $P(n)$ be the number of salamanders in year n, the difference equation is

$$P(n+1) = .85P(n) + 40$$

B. Using an initial value $P(0) = 100$, determine the populations in years $1, 2, 3, 4, 5$ according to the model you stated in part A and record the values in the following table (round any decimal values to the nearest whole number)

n	0	1	2	3	4	5
$P(n)$	100	125	146	164	180	193

C. What happens to the population in the long run? Does it tend to a definite value as n increases? If so, what is it? If not, why not?

Answer: This difference equation from part A has an equilibrium solution of

$$P(n) = \frac{40}{.15} \doteq 267,$$

since $P(n+1) - P(n) = -.15P(n) + 40 = 0$ when $P(n) = \frac{40}{.15}$. Since $.85 < 1$, this is a stable equilibrium, and the population is tending to that value as n increases. (This can be seen by writing down the general solution using the formulas for affine first order equations.)

IV. Answer *any three* of the following briefly. If you submit answers for all four, only the best three will be counted.

A. If you are fitting an *exponential* model to a data set (x_i, y_i) "by hand" (i.e. not using the shortcuts available in a Google spreadsheet) you would start by transforming the data to a new form (X_i, Y_i). What is that form in terms of the x_i and y_i?. If the best fit regression line for the *transformed data* is $Y = mX + b$, what is the corresponding exponential model? (You may use the exponential function with base 10 as we discussed in class, or any other exponential function.)

Answer: You would start out by computing the log-transformed data points $(X_i, Y_i) = (x_i, \log_{10}(y_i))$. If the regression line for the log-transformed data is $Y = mX + b$, then $\log_{10}(y) = mx + b$, so

$$y = 10^{mx+b} = (10^b) \cdot (10^m)^x$$

That is, in the exponential function $y = c \cdot a^x$, $a = 10^m$ and $c = 10^b$.

B. Give the difference equation that would model a population that (in the absence of interactions with humans) would undergo *logistic growth* with a net growth rate 15% per year when the population is much less than the carrying capacity $M = 1000$ of the habitat. Assume in addition that humans are doing constant harvesting of $h = 50$ individuals per year.

Answer: As in the Chapter 7 project, the difference equation is

$$P(n+1) = (1.15) \cdot P(n) - \left(\frac{.15}{1000}\right) \cdot (P(n))^2 - 50.$$

(The first two terms on the right are from the standard form of the logistic equation, the last is the harvesting term.)

C. (5) How do you determine *equilibrium value(s)* of a first order difference equation $Q(n+1) = AQ(n) + B$ (where A, B are constants)?

Answer: You would subtract $Q(n)$ from both sides: $Q(n+1) - Q(n) = (A-1)Q(n) + B$, then set the right side to zero and solve for $Q(n)$: $Q(n) = \frac{B}{1-A}$ (or equivalently, $\frac{-B}{A-1}$. This is the equilibrium value. (Note: this is exactly what we did in III C above.)

D. What feature of the solutions of the Lotka-Volterra equations is considered to be a confirmation that this model is capturing an important aspect of real-world predator-prey interactions?

Answer: It's the fact that *oscillations* in the predator and prey populations are observed, as in such interactions in the natural world.

Essay

In general terms, what is a mathematical model? Describe in general terms what they are, how they are constructed, and how they are used. Give examples of three different types of mathematical models we have studied this semester. Even if mathematical models don't capture *every feature* of a real world situation, why is it still important to develop them and understand the information we get from them? For instance, what conclusions about use of natural resources did we derive by looking at logistic models with various types of harvesting in the Chapter Project from Chapter 7? As another example, how are mathematical models important in understanding our choices of which energy sources to use? Why is it important to understand how radioactive substances decay? What type(s) of model(s) that we discussed would apply to describe that process? What are some of the issues involved with using radioactive decay to generate electricity–that is, why is this not a "no-brainer" as a solution to the problem of CO_2 buildup in the atmosphere from fossil fuel burning?

Model Response: A mathematical model of something is a function, graph, an equation or system of equations, etc. constructed within the "mathematical world" in order to study, or even make predictions about, the behavior of a real-world system. Constructing mathematical models relies on a process of abstraction, by which some aspects of the real-world system under consideration are not included. The predictions produced from these simplified versions of reality through use of mathematical tools are then compared with real-world data and further iterations of model construction and testing often ensue.

We have studied various types of models using linear, exponential, and power functions. We also studied single difference equations (affine and logistic equations especially) and systems of coupled difference equations (such as the Lotka-Volterra predator-prey equations) as models. In many cases, a mathematical model may be the *only* way to study a real world system where we cannot do controlled experiments. This is because it is often impossible or impractical to manipulate a natural environment for the purposes of seeing how it behaves under certain circumstances. Even when some things are "left out" of a model, if enough of the properties of the real-world system are captured, we can still get some insight or information from a model. Careful comparison with the real-world system might be needed to validate those insights or that information, though.

For instance, in looking at logistic equations with constant harvesting in the Chapter 7 project, we saw that it is possible to introduce "thresholds" at unstable equilbrium values in systems. This means that there can be harvesting levels that produce population crashes if the initial population is too small, while initial populations that are large enough yield growth toward a stable equilibrium as in the usual logistic case (with small enough r parameter values). This behavior was not present before the human intervention through the harvesting and knowing the population dynamics can work that way is an important part of sustainable management of natural resources.

Various models we studied are important in understanding our choices of energy sources. In the Chapter 3 and 4 projects, we studied how major components of our economy are based on burning of fossil fuels (petroleum, coal, and natural gas). But this is producing steadily rising atmospheric CO_2 levels that are contributing to climate change, sea level rise, and other undesirable effects. Nuclear power is sometimes considered to be a solution to some of these issues because generating electricity by using radioactive decay to produce steam for turbines does not involve any burning of fossil fuels (at least not directly). However, this source of energy has its own problems, which are revealed by considering the properties of exponential decay processes. There's always a possibility of catastrophic damage to a nuclear reactor and release of radioactivity into the environment, which would case damage to human and other populations. Even if there is no major disaster of that sort, the waste products of nuclear power remain radioactive for extended periods and hence are also dangerous for humans and other life forms. Safe long-term disposal methods for those wastes have not been developed as yet and this is another obstacle to using nuclear power for electricity generation on large scales.