College of the Holy Cross, Fall Semester 2018
MONT 104N – Modeling the Environment
Final Exam, December 13

I. The table below shows estimates collected in the EDGAR database created by the European Commission and the Netherlands Environmental Assessment Agency of the amounts of *carbon dioxide* emissions from burning of fossil fuels in 2015 and 2016 by country. The units are megatonnes $= 10^6$ metric tonnes. The populations are in units of millions $= 10^6$ of people:

Country	**CO2 2015**	**CO2 2016**	**2015 Population**
China	10.642	10.433	1367
United States	5.712	5.012	321
India	2.455	2.533	1252
Russia	1.761	1.662	142
Japan	1.253	1.240	127
Germany	.778	.776	81
Whole world	36.061	35.753	7256

A. (5) 1 metric tonne is 1000 kg and 1 kg $\doteq$ 2.205 lb. What was the equivalent amount of carbon dioxide emissions for the U.S. in 2015, in units of pounds.

Solution: The 2015 U.S. carbon dioxide emissions were

$$5.712 \times 10^6 \text{ tonnes} = 5.172 \times 10^6 \text{ tonnes} \times 1000\text{kg/tonne} \times 2.205 \text{ lb/kg} \doteq 1.259 \times 10^{10} \text{ lb}.$$

B. (5) Suppose that China's carbon dioxide emissions were decreasing *exponentially*. What would be the exponential model fitting the two data points you have exactly? Take $t = 0$ to correspond to the year 2015. What would your model predict for China's emissions in the year 2020?

Solution: The formula would be found in the form $CO2(t) = 10.642 \cdot a^t$. The value of a is determined by the value from 2016, which is $t = 1$:

$$CO2(1) = 10.433 = 10.642 \cdot a^1$$

Hence $a \doteq .98036$, so $CO2(t) = 10.642 \cdot (.98036)^t$. (This could also be done by computing the percent decrease between the 2015 value and the 2016 value.) The value in 2020, or $t = 5$, would be $CO2(5) = 10.642 \cdot (.98036)^5 \doteq 9.637$ megatonnes.

C. (10) Now, let's look at this data from another perspective. What are the *per capita* carbon dioxide emissions in units of metric tonnes per person for each of these countries? Construct a chart or charts (your choice of types) showing how the 2015 total emissions and the 2015 emissions per capita compare for these countries. Pay special attention to any changes in the relative orderings when you go from the total emissions to the per capita emissions.

Solution: We compute the per capita emissions by dividing the total amount by the population. For China, for instance the result is

$$\frac{10.642 \times 10^6 \text{ tonnes}}{1367 \times 10^6 \text{ people}} \doteq .00778 \text{ tonnes/person} = 7.78 \text{ kg/person}.$$

It's also OK to leave the answer in tonnes per person, but the numbers there will be quite small. Doing this for all of the countries and giving the answers in kg/person:

Country	**CO2 2015**	**2015 Populations**	**Per Capita**
China	10.642	1367	7.78
UnitedStates	5.712	321	17.79
India	2.455	1252	1.96
Russia	1.761	142	12.4
Japan	1.253	127	9.87
Germany	.778	81	9.605
Whole world	36.061	7256	4.97

To show this graphically, there are a number of options. One would be to give two separate bar charts, one for the total emissions levels, one for the per capita levels. If done that way, to show the different relative ordering you might list the bars in decreasing order. You could also show two bars for each country in one chart, one for the total amount, one for the per capita amount. The different units would require two different vertical scales then, so this less than optimal, but still acceptable. In any case notice the way things get reordered: In terms of total amounts, it's

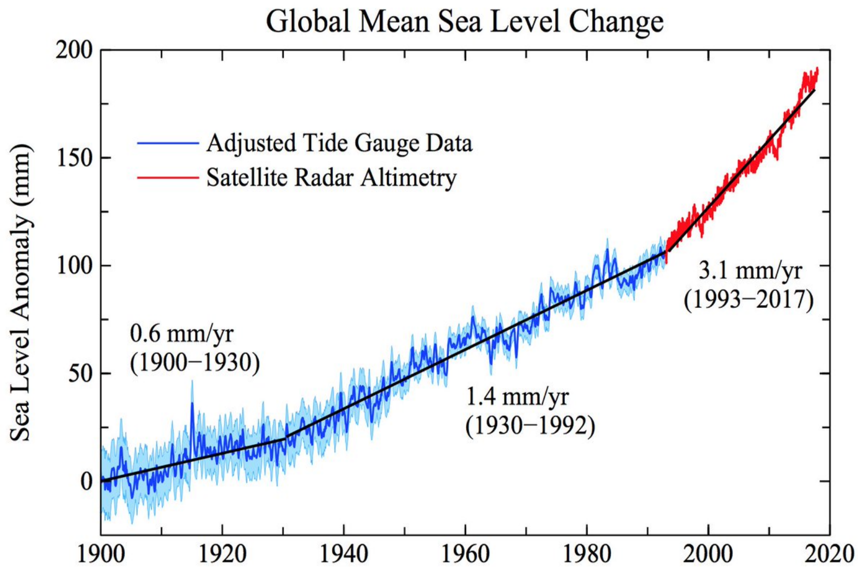
$$\text{China} > \text{US} > \text{India} > \text{Rusia} > \text{Japan} > \text{Germany}$$

But in terms of per capita amounts, it's

$$\text{US} > \text{Russia} > \text{Japan} > \text{Germany} > \text{China} > \text{India}$$

(and India is way below the world-wide per capita average). The first ordering is closely correlated with the total sizes of these nations' economies; the second ordering shows how the effects of differences in sizes of populations affect the total. The second ordering is more closely aligned with living standards and how they interact with environmental awareness.

II. The following graph constructed by Makiko Sato and James Hansen at the Columbia University Climate Science, Awareness and Solutions project, shows the change in global average sea level over the period 1900 to the present. It's also a nice example of how the basic modeling techniques we have discussed can be adapted to deal with dynamic, changing situations.



The vertical axis is the *sea level anomaly*—the *difference* between the observed levels in later years and the average level in 1900.

A. (12) The information included in the graph shows that Sato and Hansen have constructed three separate linear models over three different time periods: one covering the period 1900 - 1930, the second covering the period 1930 - 1992 and the last one covering the period 1993 to 2017. The estimated slopes are given in units of mm/yr. Find the equations of the three linear models (the equations of the three black lines in the graph) for average mean sea level $L(t)$, using $t =$ actual year in each case. Use the values 0 for $L(1900)$, 18 for $L(1930)$, and 104.8 for $L(1993)$.

Solution: The equations are (output in units of mm):

$$1900 - 1930 L(t) = .6(t - 1900) = .6t - 1140$$
$$1930 - 1992 L(t) = 1.4(t - 1930) + 18 = 1.4t - 2684$$
$$1993 - 2017 L(t) = 3.1(t - 1993) + 104.8 = 3.1t - 6073.5$$

(Note: the negative values just mean that if the lines were extended to the left back to the year 0, they would be intersecting the vertical axis way below 0. However, that is essentially a silly thing to do here, since it's extrapolating so far beyond the range of times the data came from.)

B. (3) The legend indicates that the first two models were found from adjusted tide gauge data, while the third was obtained from satellite radar measurements. Which of these two methods would you guess is more accurate? How does that relate to the appearance of the graph(s)? Discuss briefly.[1]

[1]Note: The lighter blue region around the blue line plots shows a series of "error bars" indicating a reasonable range of values consistent with the measured data. There's something like that too around the red portion of the plot, but it's much less visible.

Solution: The satellite altimetry measurements are almost certainly more accurate. This is confirmed by the fact that the lighter blue "error bars" of reasonable values extend much farther up and down from the lines than the red part of the graph does.

III. Suppose that a endangered population of spotted owls in a protected forest is *decreasing* at a net rate of 10% per year from births and deaths, but human-reared owls are being reintroduced into the habitat at a rate of 16 individuals per year.

A. (5) Write a difference equation that models the owl population.

Solution: The difference equation is

$$P(n+1) = (1 - .1)P(n) + 16 = .9P(n) + 16$$

B. (5) Using an initial value $P(0) = 40$, determine the populations in years $1, 2, 3, 4, 5$ according to the model you stated in part A and record the values in the following table (round any decimal values to the nearest whole number)

n	0	1	2	3	4	5
$P(n)$	40	52	63	73	81	89

C. (5) What happens to the population in the long run as n increase? Does it tend to a definite value? If so, what is it? If not, why not?

Solution: There is an equilibrium value at the solution of

$$P(n+1) - P(n) = -.1P(n) + 16 = 0,$$

so $P(n) = 160$. You can see this from the form of the solution with $P(0) = 40$. Using the general solution of first-order affine equations from Equation (7.4) in the text,

$$P(n) = \left(40 - \frac{16}{.1}\right) \cdot (.9)^n + \frac{16}{.1} = 160 - 120 \cdot (.9)^n.$$

Since $.9 < 1$, as $n \to \infty$, $P(n) \to 160$ and this is a stable equilibrium.

IV. Answer *any three* of the following briefly. If you submit answers for all four, only the best three will be counted.

A. (5) If you are fitting an *power function* model to a data set (x_i, y_i) "by hand," you would start by transforming the data to what new form (X_i, Y_i)?. If the best fit regression line for the transformed data is $Y = mX + b$, what is the corresponding power function model? (You may use logarithms to base 10 as we discussed in class.)

Solution: You would use a log-log transform taking the original data points to $(X_i, Y_i) = (\log(x_i), \log(y_i))$. If the regression line for the transformed data points has equation $Y = mX + b$, then to remove the logarithms we exponentiate. The corresponding power function will be obtained like this:

$$y = 10^{\log(y)} = 10^{m\log(x)+b} = 10^{\log(x^m)} \cdot 10^b = 10^b \cdot x^m.$$

The slope of the regression line becomes the scaling exponent m and 10^b gives the constant multiplier.

B. (5) What difference equation would model a population undergoing logistic growth if the population was growing at about 3% per year when the population is much smaller than the carrying capacity $M = 1000$ of the habitat?

Solution:

$$P(n+1) = 1.03P(n) - .03/1000(P(n))^2 = 1.03P(n) - .00003(P(n))^2.$$

C. (5) Describe the SIR model for infectious disease outbreaks and give the corresponding difference equations.

Solution: The population is divided into susceptible, infected, and removed subpopulations. There is no population dynamics due to births or deaths, and no immigration, or emigration take place. New infections are produced by contacts between susceptible and infected individuals, and those are modeled by product terms $S(n)I(n)$. The equations are

$$\begin{aligned} S(n+1) &= S(n) - \beta S(n)I(n) \\ I(n+1) &= I(n) + \beta S(n)I(n) - \gamma I(n) \\ R(n+1) &= R(n) + \gamma I(n). \end{aligned}$$

D. (5) What feature of the solutions of the Lotka-Volterra equations is considered to be a confirmation that this model is capturing an important aspect of real-world predator-prey interactions?

Solutions: The solutions exhibit *approximately cyclical* behavior that is similar (at least in some ways) to the behavior of real-world predator-prey systems. We discussed how this worked for the Canada lynx and snowshoe hare populations according to the famous dataset of pelts sold by trappers to the Hudson's Bay Company in Canada between 1840 and 1930.

Essay (30)

In general terms, what is a mathematical model? Describe in general terms what they are, how they are constructed, and how they are used. Give examples of three different types of mathematical models we have studied this semester. Even if mathematical models don't capture *every feature* of a real world situation, why is it still important to develop them and understand the information we get from them? For instance, what conclusions about use of natural resources did we derive by looking at logistic models with various types of harvesting in the Chapter Project from Chapter 7? As another example, how are mathematical models important in understanding our choices of which energy sources to use? Why is it important to understand how radioactive substances decay? What type(s) of model(s) that we discussed would apply to describe that process? What are some of the issues involved with using radioactive decay to generate electricity–that is, why is this not a "no-brainer" as a solution to the problem of CO_2 buildup in the atmosphere from fossil fuel burning?

Model Response: A mathematical model of something is a function, graph, an equation or system of equations, etc. constructed within the "mathematical world" in order to study, or even make predictions about, the behavior of a real-world system. Constructing mathematical models relies on a process of abstraction, by which some aspects of the real-world system under consideration are not included. The predictions produced from these simplified versions of reality through use of mathematical tools are then compared with real-world data and further iterations of model construction and testing often ensue.

We have studied various types of models using linear, exponential, and power functions. We also studied single difference equations (affine and logistic equations especially) and systems of coupled difference equations (such as the SIR and Lotka-Volterra predator-prey equations) as models. In many cases, a mathematical model may be the *only* way to study a real world system where we cannot do controlled experiments. This is because it is often impossible or impractical to manipulate a natural environment for the purposes of seeing how it behaves under certain circumstances. Even when some things are "left out" of a model, if enough of the properties of the real-world system are captured, we can still get some insight or information from a model. Careful comparison with the real-world system might be needed to validate that those insights or information are valid, though.

For instance, in looking at logistic equations with constant harvesting in the Chapter 7 project, we saw that it is possible to introduce "thresholds" at unstable equilbrium values in systems. This means that there can be harvesting levels that produce population crashes if the initial population is too small, while initial populations that are large enough yield growth toward a stable equilibrium as in the usual logistic case (with small enough r parameter values). This behavior was not present before the human intervention through the harvesting and knowing the population dynamics can work that way is an important part of sustainable management of natural resources.

Various models we studied are important in understanding our choices of energy sources. In the Chapter 3 and 4 projects, we studied how major components of our economy are based on burning of fossil fuels (petroleum, coal, and natural gas). But this is producing steadily rising atmospheric CO_2 levels that are contributing to climate change, sea level rise, and other undesirable effects. Nuclear power is sometimes considered to be a solution to some of these issues because generating electricity by using radioactive decay to produce steam for turbines does not involve any burning of fossil fuels (at least not directly). However, this source of energy has its own problems, which are revealed by considering the properties of exponential decay processes. There's always a possibility of catastrophic damage to a nuclear reactor and release of radioactivity into the environment, which would case damage to human and other populations. Even if there is no major disaster of that sort, the waste products of nuclear power remain radioactive for extended periods and hence are also dangerous for humans and other life forms. Safe long-term disposal methods for those wastes have not been developed as yet and this is another obstacle to using nuclear power for electricity generation on large scales.