

# A “case study” on Islamic mathematics

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# Outline

- 1 Goals for Today
- 2 Background
- 3 Work from the Islamic world
- 4 Later developments

# Getting here from there

- 1 Try to get some evidence regarding the claims of historians like Morris Kline that Islamic mathematics only “transmitted” Greek knowledge to Europe without adding anything very important or original
- 2 Look at a particular family of questions in number theory
- 3 Trace history from Euclid through the contributions of several Islamic mathematicians
- 4 See what we know as of know

# Back with the Pythagoreans

- 1 In early Greek mathematics the Pythagorean brotherhood
- 2 very much “into” number symbolism
- 3 the “tetraktys,” triangular numbers, squares, pentagonal numbers, ... , “perfect numbers,” “amicable pairs of numbers,” etc.
- 4 part of their philosophy of the way numbers and mathematical relationships were part of the order underlying the cosmos

## A bit of number theory from the *Elements*

- 1 In Proposition 36 of Book IX of the *Elements*, Euclid studied *perfect numbers*

### Definition

*A number  $n$  is said to be perfect if it equals the sum of all of its proper divisors (“aliquot parts” – the whole numbers less than  $n$  that divide  $n$  evenly)*

- 2 For example,  $n = 6$  is a perfect number (the smallest one) since the proper divisors of 6 are 1, 2, 3 and  $1 + 2 + 3 = 6$
- 3 Similarly  $n = 28$  is perfect since its proper divisors are 1, 2, 4, 7, 14 and  $1 + 2 + 4 + 7 + 14 = 28$

# Euclid's result

## Theorem (*Elements* IX, 36)

*A number  $n = 2^{p-1} \times (2^p - 1)$  is perfect if and only if  $2^p - 1$  is prime.*

- 1 Recall, primes are numbers  $p$  whose only divisors are 1 and the number itself:  $p = 2, 3, 5, 7, 11, 13, 17, 19, \dots$
- 2 Euclid had shown in Book IX, 20 that there are infinitely many primes (one of the jewels of this section of the *Elements*)
- 3 Not too hard to see that  $2^p - 1$  can only be prime if  $p$  itself is prime

# Perfect numbers

- 1 For example,  $2^2 - 1 = 3$ ,  $2^3 - 1 = 7$ ,  $2^5 - 1 = 31$ ,  
 $2^7 - 1 = 127$  are prime
- 2 But  $2^{11} - 1 = 2047 = 23 \cdot 89$  is not prime
- 3 So by Euclid IX, 36, for instance  $n = 2^{3-1} \times (2^3 - 1) = 28$ ,  
 $n = 2^{5-1} \times (2^5 - 1) = 496$ ,  $n = 2^{7-1} \times (2^7 - 1) = 8128$  are  
perfect numbers
- 4 But  $2^{11-1} \times (2^{11} - 1) = 2096128$  is not perfect

## Ibn al-Haytham and Thabit ibn-Qurra

- 1 Another Islamic mathematician named Ibn al-Haytham (ca. 1000 CE) (known as “Alhazen” in medieval and later Europe through a mangling of his real name!) studied this and conjectured that the *only time*  $n$  is an even perfect number is when  $n$  has Euclid’s form  $n = 2^{p-1} \times (2^p - 1)$
- 2 But he was not able to prove this completely
- 3 Thabit ibn-Qurra (836 - 901 CE), whom we met briefly last week, was interested in perfect numbers too, and other related questions

### Definition

*Two numbers  $m, n$  are said to be an amicable pair if the sum of the proper divisors of  $m$  is  $n$  and, vice versa, the sum of the proper divisors of  $n$  is  $m$ .*



## The smallest example

- 1  $m = 220$  and  $n = 284$  are the smallest amicable pair:
- 2 The proper divisors of  $m = 220 = 2^2 \times 5 \times 11$  are  
1, 2, 4, 5, 10, 20, 11, 22, 44, 55, 110 and

$$1 + 2 + 4 + 5 + 10 + 20 + 11 + 22 + 44 + 55 + 110 = 284,$$

while

- 3 the proper divisors of  $n = 284 = 2^2 \times 71$  are 1, 2, 4, 71, 142  
and

$$1 + 2 + 4 + 71 + 142 = 220.$$

## Thabit's theorem

Thabit was interested in determining whether there was a systematic way to generate amicable pairs – something along the lines of Euclid's formula for perfect numbers.

### Theorem

*Suppose that for some integer  $k \geq 1$ ,  $p = 3 \times 2^{k-1} - 1$ ,  $q = 3 \times 2^k - 1$  and  $r = 9 \times 2^{2k-1} - 1$  are all primes. Then  $m = 2^k \times p \times q$  and  $n = 2^k \times r$  are an amicable pair.*

Example:  $k = 2$  gives  $p = 5$ ,  $q = 11$ ,  $r = 71$  all prime. So  $m = 4 \times 5 \times 11 = 220$  and  $n = 4 \times 71 = 284$  are amicable(!)

## *A more balanced view(?)*

- 1 The Islamic tradition widened the range of mathematics by enlarging the role of numerical and algebraic computation and introducing the number system we still use – not just geometry
- 2 But these mathematicians also studied the foundations of Euclidean geometry and attempted to resolve some of the outstanding questions
- 3 Extended work of Archimedes and other Greeks in several ways

# Leonhard Euler

- 1 Alhazen's conjecture about Euclid's formula for perfect numbers was eventually proved true by Leonhard Euler (1707 - 1783, CE) a Swiss mathematician, one of the most prolific and original mathematicians of the modern era, built on work of Pierre de Fermat, Mersenne that *rediscovered* the results of Thabit ibn-Qurra, and others.
- 2 Might they actually have had access to Thabit's work??
- 3 Euler also generalized Thabit ibn-Qurra's method for generating amicable pairs and showed it could be used to get lots of them!

## Mathematics is not “finished!”

- 1 Students encountering mathematics in school may be tempted to think that the subject is “finished” and that everything that can be known about it is already known
- 2 But this is *far from true!!!*
- 3 The very questions we are talking about here include some very famous *unsolved problems* in mathematics – cases where no one knows the answer and where, if one of you were to be able to develop an answer, your name would join the ranks of famous mathematicians.

## For instance, ...

- 1 In Euclid's formula,  $n = 2^{p-1} \times (2^p - 1)$ , natural to ask: Are there infinitely many primes  $p$  for which  $2^p - 1$  is also a prime number? Or does this generate only finitely many different perfect numbers?
- 2 Answer: No one knows! Such primes are called "Mersenne primes" after a later French monk and mathematician Marin Mersenne (1588 - 1648, CE) who studied them
- 3 At the current time (as of March 2017), exactly 49 Mersenne primes are known to exist. The largest is  $2^{74207281} - 1$
- 4 Found via extensive computer calculations including massive computations on distributed networks of computers over the internet – GIMPS project: can donate your computer's free time to the search if you want!

## For instance, ...

- 1 In Euclid's formula for perfect numbers,  $n = 2^{p-1} \times (2^p - 1)$ , is always *even*, so when  $n$  is perfect, we only get even perfect numbers this way.
- 2 Question: Are there any *odd* perfect numbers?
- 3 Answer: Nobody knows!!
- 4 Some indications that the answer is probably "no," but nothing definitive yet. Any progress here would basically make a mathematician's career!

## For instance, ...

- 1 Similarly, one can ask: Are there infinitely many amicable pairs?
- 2 Answer: Nobody knows!!
- 3 By using Thabit's formulas, Euler's generalizations, lots of computing, many such pairs are known (in the millions!) but again, nothing definitive.