MONT 107Q – Thinking About Mathematics
Problem Set 2, **due:** 5:00pm on Monday, March 13.
More on Diophantus, *Arithmetica*

I. Here is a more-or-less literal translation of Diophantus' Proposition 12 from Book I:

Book I, Proposition 12. To divide a given number into two parts *twice*, so that one of the numbers in the first decomposition has a given ratio to one of the numbers in the second decomposition, and the other number in the first decomposition has another given ratio to the other number in the second decomposition.

Let it be proposed to divide 100 into two parts twice, so that the larger number in the first decomposition is in 2^{pl} ratio with the smaller number in the second decomposition and the larger number in the second decomposition is in 3^{pl} ratio with the smaller number in the first decomposition.

Let the smaller number in the second decomposition be set out as $\varsigma 1$ then the larger number in the first decomposition will be $\varsigma 2$. The smaller number in the first subdivision is then $M^o 100 \Lambda \varsigma 2$. And since the triple of this number is the larger number in the second subdivision, that number is $M^o 300 \Lambda \varsigma 6$. But the sum of $\varsigma 1$ and $M^o 300 \Lambda \varsigma 6$ is 100. So $M^o 100$ is $M^o 300 \Lambda \varsigma 5$. Therefore ς is 40.

To what was set down.

- The larger number in the first decomposition is $\varsigma 2$, which is $M^o 80$.
- The smaller number in the first decomposition is $M^o 100 \Lambda \varsigma 2$ which is $M^o 20$.
- The triple of the smaller number in the first decomposition is $M^o 60$ (This is the larger number in the second decomposition.)
- The smaller number in the second decomposition is ς, which is $M^o 40$.

And the verification is evident.

A) Translate Diophantus' proof into modern algebra and show that his method is correct.

B) Show how the method given here could be generalized if the given number is any n, the two decompositions are $n = x_1 + x_2$ and $n = x_3 + x_4$ with $x_1 > x_2$ and $x_3 > x_4$ and we are given the ratios $x_1 = ax_4$ and $x_3 = bx_2$ with a, b some positive integers.

II. Now let's go on and look a bit at a problem in one of the later books of the *Arithmetica*. Here's Proposition 1 in Book II:

Book II, Proposition 1. To find two numbers whose sum has a given ratio to the sum of their squares.

Let it be proposed that the sum of the squares is in 10^{pl} ratio with the sum. (Comment: Diophantus actually states this the opposite way saying the sum is the 10th part of the sum of the squares.) Further let it be proposed that the smaller of the two numbers is ς

and the larger is $\varsigma 2$. Then the sum is $\varsigma 3$ and the sum of the squares is $\Delta^{\Upsilon}5$. We must have that $\Delta^{\Upsilon}5$ and $\varsigma 30$ are the same. So ς is $M^o 6$. The other number is $M^o 12$ and the problem is solved. [Note: As Diophantus progresses through the *Arithmetica*, he writes less and less; presumably his student doesn't need as much detail as he/she progresses in the subject.]

A) Translate Diophantus' proof into modern algebra and show that his method is correct for these numbers. (That is show that the sum of the squares of the two numbers is 10 times the sum of the two numbers.) In particular, be sure to explain how you are interpreting the relation between ς and Δ^{Υ}. How are those numbers connected?

B) The big question in your mind should be: where did the assumption "the larger is $\varsigma 2$" come from? What would happen if the two numbers were in some other ratio? Say the larger is ςa for some given number a? Can you still solve the problem? Can you always solve it with whole numbers?