MONT 106Q – Mathematical Thinking
Problem Set 2 – Egyptian Mathematics – Selected Solutiosn
September 27, 2016

I. *Egyptian multplication and division.*

C) Divide by the Egyptian method: $167 \div 18$.

Solution: We want to calculate with 18 to yield 167, so we start by doubling:

$$\begin{aligned} 1 \times 18 &= 18 \\ 2 \times 18 &= 36 \\ 4 \times 18 &= 72 \\ 8 \times 18 &= 144 \end{aligned}$$

Using these we can make $9 \times 18 = 144 + 18 = 162$. The left over part (remainder) is $167 - 162 = 5$. Now to get this fractional part of the quotient in an easy way, recall that the Egyptians did allow the non-unit fraction 2/3. This would be useful here, for instance if we start by multiplying by 2/3, then multiply by 1/2 once and 2/3 again:

$$\begin{aligned} 2/3 \times 18 &= 12 \\ 1/3 \times 18 &= 6 \\ 1/6 \times 18 &= 3 \\ 1/9 \times 18 &= 2 \end{aligned}$$

So we get $167 \div 18 = 9 + \frac{1}{6} + \frac{1}{9}$. (This was not really an exact science for the Egyptians – it seems as though they were happy if they just found something that worked!)

II. *Egyptian fractions.* A part of the table of conversions of $2/n$ into unit fractions from the Rhind papyrus is given in the handout from class on Wednesday, September 21. Each row gives an expansion

$$\frac{2}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

for one of the odd integers $3 \le n \le 101$. (if there is no entry for c and/or d, then those terms are not present).

A) Verify that the rows for $n = 13$ and $n = 29$ are

Solution: What I meant for you to do on this one was something like this: From the table $\frac{2}{13} = \frac{1}{8} + \frac{1}{52} + \frac{1}{104}$. But putting the fractions over a common denominator and adding,

$$\frac{1}{8} + \frac{1}{52} + \frac{1}{104} = \frac{13 + 2 + 1}{104} = \frac{2 \cdot 8}{13 \cdot 8} = \frac{2}{13}.$$

This verifies that the entry is *exactly* correct. With a finite decimal approximation, the most you can say is that the results are close. (The decimal expansion for 2/13 is infinite and repeating.) The other one is similar.

B) Use the table to find a decomposition of $\frac{4}{27}$ into unit fractions.

Solution: From the table

$$\frac{2}{27} = \frac{1}{18} + \frac{1}{54}$$

so doubling and canceling factors of 2:

$$\frac{4}{27} = \frac{2}{18} + \frac{2}{54} = \frac{1}{9} + \frac{1}{27}.$$

(There are other correct ways to do this also.)

C) Show that for any positive integer n

$$\frac{2}{n} = \frac{1}{n} + \frac{1}{2n} + \frac{1}{3n} + \frac{1}{6n}.$$

Yet this expansion appears only once in the table from the Rhind papyrus. For which n did the compiler of the table fall back on this form? Can you see why they might have preferred to avoid using this if possible?

Solution: The formula is valid for general n because if we find the common denominator $6n$, then

$$\frac{1}{n} + \frac{1}{2n} + \frac{1}{3n} + \frac{1}{6n} = \frac{6+3+2+1}{6n} = \frac{12}{6n} = \frac{2}{n}.$$

It's not enough to check that this works for one (or any finite number of) specific value(s) of n because the question is asking for a general proof that the formula *always works.* This form appears in the Rhind papyrus $2/n$ table only for $n = 101$. It is likely that the Egyptians prefered to use sums with 2 or 3 terms when they could find them to simplify calculations.

IV. *An Egyptian approximation to π.* We take π to be the ratio between the area of a circle and the square of the radius: $\pi = \frac{A}{r^2}$.

A) In the Rhind papyrus, the area of a circle with diameter d is given as

$$A_{circ} = \left(\frac{8}{9}d\right)^2.$$

(This is not exactly correct, but yields decent approximate results.) What is the implied approximation to the value of π here?

Solution: The Egyptian formula would be correct if

$$\left(\frac{8d}{9}\right)^2 = \pi\left(\frac{d}{2}\right)^2$$

which would say $\pi = \frac{4 \cdot 64}{81} \doteq 3.16$. That is quite close to the exact value of $3.14159\cdots$.

B) Here's one way this may have been derived. Take a square of side 9 units, divide the edges into three equal pieces, and cut off four corner triangles with side length 3 to

make an octagon. The octagon is pretty close in area to the circle inscribed in the original square. Explain why the octagon has area $63 \doteq 8^2$, and then explain how the approximation $A_{circ} = \left(\frac{8}{9}d\right)^2$ would result.
Solution/Comment: If we inscribe a circle of radius d in a square of side d, the area is very close to the area of the octagon obtained by cutting off the four corners of the square as described. This octagon has exact area $\frac{7d^2}{9}$. But then

$$\frac{7d^2}{9} = \frac{63d^2}{81} \doteq \frac{64d^2}{81} = \left(\frac{8d}{9}\right)^2.$$

This is thought to be how the Egyptians derived this approximate area formula for the circle.