MONT 105Q – Mathematical Journeys
Midterm Exam Solutions, April 1, 2016

I. The following data set has $n = 11$:

$$1.3,\ 2.1,\ 0.8,\ 2.3,\ 3.9,\ 4.1,\ 2.2,\ 1.7,\ 1.2,\ 3.1,\ 2.4$$

(Note: These are *not in numerical order.*)

(A) (5) Find the "5-number" summary for this data set (min, 1st quartile, median, 3rd quartile, max).

Solution: After reordering smallest to largest we have:

$$0.8,\ 1.2,\ 1.3,\ 1.7,\ 2.1,\ 2.2,\ 2.3,\ 2.4,\ 3.1,\ 3.9,\ 4.1.$$

Then

$$\begin{aligned} \text{minimum} &= 0.8 \\ \text{first quartile} &= 1.3 \\ \text{median} &= 2.2 \\ \text{third quartile} &= 3.1 \\ \text{max} &= 4.1 \end{aligned}$$

(B) (5) Construct a frequency histogram for the data using 5 equal bins on the range 0.0 to 5.0.

Solution: The bins and counts (= heights of the boxes) are

$$[0,1) \to 1, [1,2) \to 3, [2,3) \to 4, [3,4) \to 2, [4,5] \to 1$$

(C) (10) Compute the SD of the data set, showing all work to justify your answer. How many of the data values lie within one SD of the mean?

Solution: We compute the mean $\overline{X}$ first:

$$\overline{X} = \frac{1.3 + 2.1 + 0.8 + 2.3 + 3.9 + 4.1 + 2.2 + 1.7 + 1.2 + 3.1 + 2.4}{11} \doteq 2.2818$$

Then the SD is

$$\sqrt{\frac{1.3^2 + 2.1^2 + 0.8^2 + 2.3^2 + 3.9^2 + 4.1^2 + 2.2^2 + 1.7^2 + 1.2^2 + 3.1^2 + 2.4^2}{11} - (2.2818)^2} \doteq 1.0142.$$

The range within 1 SD of the mean is $2.2818 - 1.0143 = 1.2675$ to $2.2818 + 1.0143 = 3.2961$. Of the 11 numbers, 7 are in this range. (Comment: $7/11 \doteq .64$, which fits well with the "empirical rule" based on the normal distribution.)

(D) (5) If the numbers in this data set are written on slips of paper and put in a box, the box is shaken, and two slips are drawn at random without replacement, what is the probability that both are < 3.0?

Solution: There are 8 numbers < 3.0 to begin with. After a first number < 3.0 is chosen, 7 of the remaining 10 numbers are < 3.0. So this probability is $\frac{8}{11} \cdot \frac{7}{10} = \frac{56}{110} = \frac{28}{55} \doteq .51$. (This could also be computed as the ratio $\frac{\binom{8}{2}}{\binom{11}{2}}$.)

II. Answer each question with a sentence or two. Suppose that a researcher collects 40 tufted titmice (these are small North American songbirds) to measure their body weights. The collected birds have an average weight of 25 g with an SD of 1 g. Think of this as a sampling process.

(A) (5) What is the population? What is the sample?

Solution: The population would be the weights of all tufted titmice. The sample is the 40 weights of the collected birds.

(B) (5) What could we say about the distribution of sample mean weights of samples of size 100 of tufted titmice (even if the distribution of the individual weights was unknown)? Explain.

Solution: By the Central Limit Theorem, the sample mean weights would be approximately normally distributed with mean equal to the population mean weight and SD equal to the population SD divided by $\sqrt{100}$. This last quantity could be estimated by $\frac{1}{\sqrt{100}} = .1$.

(C) (5) The average lifespan of tufted titmice is 2.1 years but many birds die as nestlings (i.e. before reaching maturity, which takes about 4 months from the time they hatch). Would you expect the average age of birds in a sample of adults collected in the wild to be larger or smaller than 2.1 years? Explain.

Solution: Given that we are observing living, adult birds, they have survived the nestling stage, so the average age should be *greater than* 2.1 years. (Another way to say this: the birds that die as nestlings pull the average age down. So the ages of the living adult birds we collect would be larger.)

(D) (5) If you knew that the researcher was only taking birds from a particular location, is that a random sample from the appropriate population? Might that process introduce a bias in the measured weights?

Solution: It is probably not a random sample, and it could bias the measured weights if food is particularly plentiful (or not) in the area where the sample is taken.

III. Suppose that a large data set of standardized raw test scores is *normally distributed* with $\mu = 75$ and $\sigma = 4$. The "z-score" of a raw test score x is computed by

$$z = \frac{x - \mu}{\sigma}.$$

(A) (5) What would be the z-score of a raw test score of 80?

Solution: $z = \frac{80-75}{4} = 1.25$.

(B) (5) Based on this information, if a raw test score x is selected at random from the data set, what is the probability that $74 \le x \le 80$?

Solution: $P(74 \le x \le 80) = P(-.25 \le z \le 1.25)$. Using the normal table, this is: $(.5 - .4013) + (.5 - .1056) = .4931$.

(C) (10) Based on this information, if $n = 20$ raw test scores are selected at random, what is the probability that *none of them* is in the range $74 \le x \le 80$.

Solution: This is

$$(1 - .4931)^{20} = \binom{20}{0}(.4931)^0(1 - .4931)^{20} \doteq .000001$$

(i.e. virtually no chance of this happening!)

IV. Essay. (35) What is the central thesis of the article *The Dawning of the Age of Stochasticity* by David Mumford? How does Mumford think mathematics should develop in the current century? How is his proposal related to the outcomes of the various attempts to solve the "crisis of foundations" we read about in *Logicomix*? Would Mumford agree or disagree with this statement: "what mathematicians should be doing is closer to the kind of science Darwin practiced on the voyage of the *Beagle* than it is to a Euclidean search for absolute certainty." Explain.

Model Response: The central thesis of Mumford's article is that stochastic (that is, probability-based) models and statistical thinking will be more important in the mathematics of the 21st century than exact models, logical argumentation and proof. They are more relevant to the real world, to many parts of mathematics, and in particular to understanding how our minds work. In a way, we might say this suggestion is at least in part a reaction to the fact that none of the various attempts to put mathematics on a completely secure footing and solve the "crisis in foundations" really got off the ground. They never successfully addressed the basic concerns about the certainty of mathematical knowledge. Mumford points out (rather scornfully, too) how the logicist approach taken in Russell and Whitehead's monumental *Principia Mathematica* only succeeded in developing the most basic of mathematical facts (e.g. $1 + 1 = 2$), at the cost of hundreds of pages of very hard work and almost impenetrable symbolism. Similarly, Hilbert's dream of axiomatic systems powerful enough to allow all true statements to be proved and all false statements to be proved false was effectively dashed by Gödel's incompleteness theorems. The constructivist approaches to mathematics have been marginally more successful. But very few mathematicians work that way (restricting themselves to finite sets and things that can be constructed in finite terms). Constructive methods still require some very hard work to prove very basic facts, especially in areas like calculus. If Mumford's ideas become widely accepted, the net effect of his polemic in favor of stochastic models and statistical thinking would definitely be to make mathematics *closer to the other sciences*, and closer to what Darwin was doing in his observations. That would give a greater role to the sort of inductive reasoning that scientists also do and a smaller role to deductive reasoning and logical argumentation. This sort of work will of necessity lack the sort of absolute certainty that people ascribed to the results of Euclidean geometry. So he *would agree with this statement.*