MONT105Q – Mathematical Journeys
Selected Solutions for Problem Set 2
due: Friday, March 4, 2016

I. Four cards will be dealt off the top of a well-shuffled deck (without replacement, of course!).

(A) Which is more likely, or are they equally likely? Explain and/or calculate probabilities:

(i) The cards are a spade, then a heart, then a diamond, then a club, or

(ii) the cards belong to four different suits?

Solution: It's not necessary to compute anything here (although one certainly *can do so*). To see that it is more likely that the cards belong to four different suits, just note that any collections of cards as in (i) also satisfy the condition in (ii). Since the four suits could come in any order, then in fact (ii) will be exactly $24 = 4!$ times as likely as (i).

(B) Again, which is more likely and explain:

(i) The cards all belong to the same suit, or

(ii) the cards belong to four different suits?

Solution: Here, some computation is probably necessary (unless you have really good intuition). Let's also look at two different ways to do the counting here which illustrate some good techniques to have at one's disposal. First, consider (i) – the cards all belong to the same suit.

- Method 1: Consider the four cards ordered as they are dealt out. There are 52 choices for the first card, then the other cards have to come from the suit of the first card, so 12 choices for the second, 11 for the third and 10 for the fourth. The total number of ordered lists is the number of permutations of 52 things taken 4 at a time, or $52 \cdot 51 \cdot 50 \cdot 49$. This makes the probability of getting four cards in the same suit
$$\frac{52 \cdot 12 \cdot 11 \cdot 10}{52 \cdot 51 \cdot 50 \cdot 49} \doteq .01$$
- Method 2: As unordered lists (combinations), there are four suits, $\binom{13}{4}$ unordered sets of four cards in each suit, and $\binom{52}{4}$ different four-card hands. The probability computed this way is
$$\frac{4 \cdot \binom{13}{4}}{\binom{52}{4}} = \frac{\frac{4 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{4 \cdot 3 \cdot 2 \cdot 1}}{\frac{52 \cdot 51 \cdot 50 \cdot 49}{4 \cdot 3 \cdot 2 \cdot 1}}$$

which is the same. *The reason the first method gives the same result is that every combination occurs the same number of times* ($4! = 24$) *in all possible different orderings.* The denominators of 4! cancel in Method 2, giving the same result as in Method 1.

Next consider counting problem (ii) – the probability of getting cards in four different suits. This can also be done the two different ways discussed above.

- Method 1: Consider the four cards ordered as they are dealt out. There are 52 choices for the first card, then the second card has to come from one of the other three suits (39 possiblities), then $2 \cdot 13 = 26$ choices for the third and 13 for the fourth. The total number of ordered lists is the number of permutations of 52 things taken 4 at a time, or $52 \cdot 51 \cdot 50 \cdot 49$. This makes the probability of getting four cards in the same suit

$$\frac{52 \cdot 39 \cdot 26 \cdot 13}{52 \cdot 51 \cdot 50 \cdot 49} \doteq .1$$

- Method 2: As unordered lists (combinations), there are four suits, $\binom{13}{1}$ choices of a card in each suit. As before there are $\binom{52}{4}$ different four-card hands. The probability computed this way is

$$\frac{13^4}{\binom{52}{4}} = \frac{(13 \cdot 13 \cdot 13 \cdot 13)(4 \cdot 3 \cdot 2 \cdot 1)}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{52 \cdot 39 \cdot 26 \cdot 13}{52 \cdot 51 \cdot 50 \cdot 49}$$

which is the same. *Once again, the reason the first method gives the same result is that every combination occurs the same number of times* ($4! = 24$) *in all possible different orderings.* The denominators of 4! cancel in Method 2, giving the same result as in Method 1.

II. One ticket will be drawn at random from a box containing 6 tickets: three with the numbers 2,2,3 on blue paper, and three with the numbers 2,2,3 on green paper. Are the number and the color independent? Why or why not?

Solution: They *are* independent. The reason is that, for instance

$$P(2|\text{green}) = \frac{2}{3} = P(2|\text{blue})$$

but $P(2) = \frac{4}{6} = \frac{2}{3}$ as well Similarly $P(3) = \frac{1}{3}$ which is the same as $P(3|\text{green})$ and $P(3|\text{blue})$.

III. A fair, standard die is rolled 10 times. Find the probability of

(A) getting 10 sixes

Answer: $\frac{1}{6^{10}}$

(B) not getting 10 sixes (that is, anothing other than 10 sixes)

Answer: $1 - \frac{1}{6^{10}}$

(C) all rolls showing four or fewer spots

Answer: $(\frac{4}{6})^{10}$

(D) the sum of the numbers shown in the 10 rolls is greater than or equal to 57.

Solution: For this one, we need to think how many ways there are of obtaining rolls adding up to $57, 58, 59, 60$.

- 60 is easy – the only way this happens is if all 10 rolls give a six – this gives 1 possible ordered list: $6, 6, 6, 6, 6, 6, 6, 6, 6, 6$.
- 59 is only obtainable as 9 sixes and 1 five. The five can occur in any one of the 10 positions in the list, so there are $\binom{10}{1} = 10$ different ordered lists here.
- 58 can only be obtained as 9 sixes and 1 four, or as 8 sixes and two fives. There are $\binom{10}{1} = 10$ of the first ones, and $\binom{10}{2} = 45$ of the second.
- 57 can only be obtained as 9 sixes and 3 four, as 8 sixes a five and a four, or as 7 sixes and 3 fives. There are $\binom{10}{1} = 10$ of the first ones, and $2\binom{10}{2} = 90$ of the second (note that for every choice of a pair of "slots" in the 10 rolls, there are still 2 different ways to place the four and the five). Finally, there are $\binom{10}{3} = 120$ of the last type.

In all, this makes the probability of getting a total no smaller than 57 is

$$\frac{1 + 10 + 10 + 45 + 10 + 90 + 120}{6^{10}} = \frac{286}{6^{10}} \doteq .000005$$

V. A fair coin will be tossed 10 times. What is the probability that exactly 2 of the first 5 tosses will yield heads, and exactly 3 of the last 5 tosses will yield heads? What are you assuming here? Also, *without calculations, explain* whether your answer is less than, greater than, or equal to the probability that there are exactly 5 heads in the 10 tosses.

Solution: Using the binomial formula, the probability that there are exactly 2 heads in the first 5 tosses is $\binom{5}{2}(.5)^2(.5)^3 = \frac{10}{32} = \frac{5}{16}$ and similarly the probability that there are exactly 3 heads in the last 5 tosses is $\binom{5}{3}(.5)^3(.5)^2 =$

$\frac{10}{32} = \frac{5}{16}$. The probability that there are exactly 2 heads in the first 5 *and* exactly three heads in the last 5 is

$$\frac{10}{32} \cdot \frac{10}{32} = \frac{100}{1024}.$$

This is definitely smaller than the probability that there are exactly 5 heads overall: $\binom{10}{5}(.5)^5(.5)^5 = \frac{252}{1024}$. Note also that the ways to get exactly 5 heads overall *contain* cases where there are exactly 2 heads in the first five and 3 heads in the second five, but there are other ways it can happen too. This gives another intuitive way to see that the probability of exactly five heads is larger.