MONT 104Q – Mathematical Journeys
Solutions for Midterm Exam, November 2, 2015

I.

(A) (10) G. H. Hardy included two theorems and proofs from Euclid's *Elements* as prime examples of "real, serious, beautiful mathematics" in his book *A Mathematician's Apology*. Give the statements of those two theorems. *You do not need to supply the proofs for these.*

First theorem: *Solution:* There are infinitely many prime integers.

Second theorem: *Solution:* The number $\sqrt{2}$ is irrational (that is, it cannot be written in the form $\sqrt{2} = \frac{a}{b}$ where a, b are integers).

(B) (5) Hardy says "the mathematics which has permanent aesthetic value ... may continue to cause intense emotional satisfaction to thousands of people after thousands of years." About how much time (to within 100 years) separates the time of Euclid from Hardy's own time?

Solution: It was about 2200 years, give or take. (Euclid lived about 300 BCE and Hardy was alive 1877 - 1947 CE, so taking 1900 CE as a date for Hardy, that's about 2200 years.)

II.

(A) (10) State the 5 Common Notions (Axioms) and 5 Postulates at the start of Book I of the *Elements* of Euclid.

Solution: The Common Notions (This is just one possible way to state all of these. I'll accept alternates meaning the same, and they don't need to be in this order)

(1) Things that are equal to the same thing are equal to one another.
(2) If equals are added to equals the wholes are equal.
(3) If equals are subtracted from equals, the remainders are equal.
(4) Things that coincide with one another are equal to one another.
(5) The whole is greater than the part.

The Postulates: (This is just one possible way to state all of these. I'll accept alternates meaning the same.)

(1) (It is possible) to draw a straight line from any one point to any other point.
(2) (It is possible) to extend a line indefinitely in both directions.
(3) (It is possible) to construct a circle with any given point as center and radius equal to any given line segment.
(4) All right angles are equal.
(5) If a line falling on two other lines makes angles on one side totaling less than two right angles, then the two lines, when extended indefinitely, meet on the side on which the two angles totaling less than two right angles are located.

(B) (5) What is the purpose of the statements from part A in Euclid's logical scheme? How are the Common Notions different from the Postulates?

Solution: The purpose of these statements is to give a starting point for reasoning about geometry. These statements are understood as unproved starting assumptions and the basic facts of plane geometry are deduced from these. The Common Notions are different from the Postulates in that they (the Common Notions) are more general – they are principles of quantitative reasoning in general. The Postulates are more specfically statements about geometry.

III. (10) Proposition 1 in Book I of Euclid's *Elements* is a construction for an equilateral triangle with side equal to any given line segment AB. Give the construction and the proof that the construction is valid, with justifications for each of the steps in the proof based on your answers to question II.

Solution: The construction: Given: the segment AB.

(1) With center A and radius AB, form one circle (Postulate 3).

(2) With center B and radius AB, form a second circle (Postulate 3).

(3) Let C be one of the intersections of the two circles and connect AC and BC with line segments (Postulate 1).

Proof: The claim is that ΔABC is equilateral. First, $AB = AC$ since they are both radii of the first circle. Similarly, $AB = BC$ because they are both radii of the second circle. Hence $AC = BC$ as well by Common Notion 1. Therefore ΔABC is equilateral.

IV. Proposition 47 in Book I of the *Elements* is a famous statement from geometry illustrated by the figure above. Use the labeling here in your answers to all parts.

(A) (5) Give the statement *in Euclid's form* and the usual name of this result.

Solution: Euclid's statement is: Let ΔABC be a right triangle with right angle at A. Then the square on the side BC opposite the right angle (the hypotenuse) is equal in area to the sum of the squares on the sides AB and AC. This is the *Pythagorean Theorem*, but stated in terms of areas.

(B) (5) How is the dotted line AM in the figure constructed?

Solution: It's constructed to pass through A and be *parallel* to the line containing BD. The construction for that is given in a previous proposition (Proposition 31, to be exact).

(C) (5) In the first part of the proof, Euclid shows that ΔGBF has the same area as what other triangle in the figure? Why does that follow?

Solution: ΔGBF has the same area as ΔCBF. This follows because those two triangles have the same base and are in the same parallels (Proposition 37). Euclid establishes that CG and FB lie on parallel lines by considering the alternate interior angles for the transversal line containing AB.

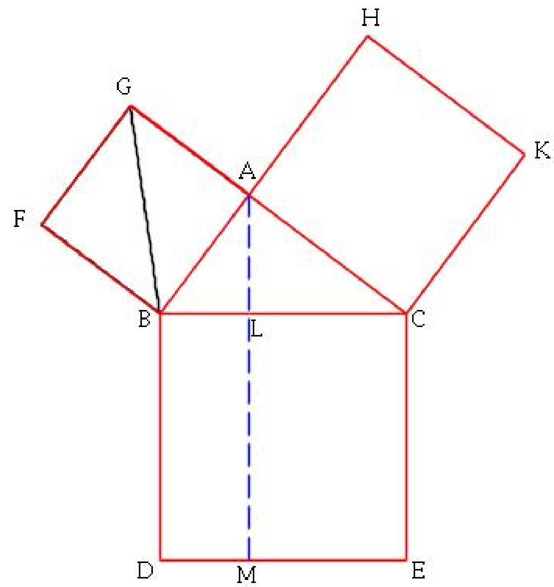

Figure 1: Figure for Proposition 47, Book I

(D) (5) The second part of the proof consists of showing that ΔFBC and ΔABD are congruent. How does that follow? (Show that is true using one of the triangle congruence results proved before in Book I.)

Solution: We have $FB = AB$ since they are two sides of the same square. Similarly $BC = BD$ since they are two sides of the same square. Finally, $\angle FBC = \angle FBA + \angle ABC = \angle ABC + \angle CBD = \angle ABD$, where the middle equality uses the facts that $\angle FBA$ and $\angle CBD$ are both right angles and Postulate 4. Then ΔFBC and ΔABD are congruent by the SAS congruence criterion (Proposition 4).

(E) (5) How does Euclid conclude that $ABFG$ and $BLMD$ have the same area? And how does he conclude the proof?

Solution: First he shows that ΔABD and ΔBDM have the same area using Proposition 37 again. Then Common Notion 1 says ΔGBF and ΔBDM have the same area. But ΔGBF has half the area of the square $ABFG$ and ΔBDM has half the area of the rectangle $BLMD$. So $ABFG$ and $BLMD$ also have the same area (Common Notion 2). Euclid concludes the proof by saying that a similar argument shows the area of the other square $ACKH$ is equal to the area of the rectangle $LMEC$. Then adding we get that the area of the square $BDEC$ is equal to the sum of the areas of the squares $ABFG$ and $ACKH$.

IV. Essay. (35) George G. Joseph, the author of an interesting book about the history of mathematics called *The Crest of the Peacock*, offered this overall evaluation of the ultimate impact of Greek geometry: "There is no denying that the Greek approach to mathematics produced remarkable results, but it also hampered the subsequent development of the subject. ... Great minds such as Pythagoras, Euclid, and Apollonius spent much of their time creating what were essentially abstract idealized constructs; how they arrived at a conclusion was in some way more important than any practical significance." First, what does the last sentence mean? What is Joseph getting at? Then, based on what he said in *A Mathematicians Apology*, how would G.H. Hardy respond to Joseph? Finally, which side of this debate do you come down on personally? Should all the mathematics we learn and do have practical usefulness or significance?

Sample Response:

The last sentence in the quote from Joseph's book is referring to the fact that in texts like Euclid's *Elements*, much attention is paid to the abstract geometric relationships of triangles, parallelograms, squares, etc. and to the logical sequence of the postulates and propositions. But as we have seen Euclid devotes no attention whatsoever to possible uses of geometry to solve real-world problems. Euclid seems in particular to care more about how statements are proved ("how they arrived at a conclusion") than in potential applications or practical consequences. Moreover, Joseph is saying that this focus on proof and on the logical structure of mathematical deduction *held Greek and later mathematics back* because it divorced mathematics from the applications that would have lead to other new ideas and discoveries. So that aspect of Greek mathematics was not, ultimately, healthy for the subject, according to Joseph. [*Comment:* For Joseph, and for many modern mathematicians (but not Hardy!), a lot of really interesting mathematics comes from the practical problems and questions from the sciences and other areas of inquiry.]

Based on the ideas he expressed in *A Mathematician's Apology*, we can safely assume that Hardy would have *disagreed vehemently* with Joseph. For Hardy, any "practical significance" of results is definitely of secondary importance. He is more interested in the logic of proofs and the sense of beauty and unexpectedness he finds in "serious, real" mathematics like the propositions from Euclid that he quotes as his prime examples (from question I before). Joseph would say those deal with "abstract, idealized constructs," while Hardy would say in effect that that is why they are interesting to him. Indeed, Hardy finds most of applied mathematics ugly and boring. While he acknowledges that mathematics can be useful, it is not that side of the subject that appeals to him.

[Any personal opinion is OK for the last part!]