MONT 104Q – Mathematical Journeys
Information for Final Exam
December 7, 2015

General Information

The final exam announced in the course syllabus will be given at 11:30am on Friday, December 18 in our regular classroom, Swords 209. You will have the entire 2.5 hour exam period to work on the exam if you want or need that much time. However, if you are well-prepared and work steadily, I do not expect that it will take any longer than 1.5 hours to complete.

Miscellaneous Groundrules

No use of cell phones, pagers, I-pods, I-pads, or any other electronic devices will be allowed during the exam. There will not be any numerical computations requiring use of a calculator. You will be asked to turn all devices off and stow them in your backpack.

What Will Be Covered

The exam will cover the mathematical material we have studied since the start of the semester, but *not* the CHQ common readings (except for one option on the essay portion of the final – see below).

- Our discussion of Hardy's *A Mathematician's Apology*, Hardy's ideas about pure and applied mathematics, his sense of mathematical beauty (esthetics), his thoughts about his own life and why he did what he did
- Book I of the *Elements* of Euclid
- Our proof of the "power theorem" (which said that if m is a prime number and $1 \leq r \leq m-1$, then $r^{m-1} R m = 1$). Incidentally, the "official name" of this result is *Fermat's "Little Theorem"*. (It's called that to distinguish it from Fermat's "Last Theorem" which was an unsolved problem until about 1994, when Andrew Wiles and Richard Taylor completed a proof).
- Imre Lakatos' ideas about the development of mathematics through proofs and refutations and how he essentially rules out the idea of absolute certainty derived from proofs of mathematical statements. (You're free to disagree with that, of course; many mathematicians would also disagree. But you should understand that that *is* Lakatos' point of view, at least about the current frontiers of mathematics, the things mathematical researchers are actually working on at any given time.)
- The relation $V - E + F = 2$ for convex polyhedra as developed through our reading of the first sections of *Proofs and Refutations*.

This means in particular that I might ask you about

1. The statements *and proofs* of the two theorems Hardy gives as supreme examples of "real mathematics" (from the first group discussion day). (A word to the wise: We

did not do the proofs on the midterm; you should expect to see one of the proofs on the final.)

2. Know the statements and the proofs of Propositions 27, 28, 29 and 32 of Book I of the *Elements*. Be able to identify what Axioms (Common Notions), Postulates, and/or previously proved results are used in the proofs of these, how Propositions 27 and 29 are related, and know especially why Proposition 29 is such an important step in the development of Book I.
3. Know the statement and the proof of Proposition 47 in Book I of the *Elements*. For this one, you needn't memorize the numbers of the previous Propositions that are used in the proof (because there are quite a few). It will suffice just to say "by a previous proposition, we know ... ." (Note: This appeared on the midterm exam too, but it's very important and a high point of what we have done!)
4. Know the pattern that we found in the remainders $r^{m-1}Rm$ for m a prime number and $1 \leq r \leq m-1$. Also know the proof we developed that this always holds.
5. Know the relation $V - E + F = 2$ for convex (and other) polyhedra and how we sketched a proof in that case. (Recall, this was an unsatisfyingly incomplete "monster-barring" exercise for Lakatos, but it is an interesting proof even if it is not "the whole story" about when $V - E + F = 2$ holds for polyhedra.)

Format

Approximately 65% of the exam will consist of questions related to the mathematical results and theorems above. The remaining approximately 35% will be an essay (target length: about 2 handwritten pages), on one of the topics below. To prepare well, you will want to think (and probably write) out practice essays on the topics. In the evaluation of the exam essay, I will be looking at the content and organization of what you are saying, but *not* at mechanical issues of grammar, punctuation, spelling, etc.

The Essay Question

For the essay question, you will have a *choice* of either option A or option B below.

- Option A: Write on *one of the following topics* (yes, one of these exact questions). I will choose which of these two appears on the exam.

1) Oliver Heaviside, 1850-1925, an English engineer, applied mathematician, and physicist, once wrote the following about the role of Euclid in mathematical education in his time in England: "As to the need of improvement there can be no question whilst the reign of Euclid continues. My own idea of a useful course is to begin with arithmetic, and then not Euclid but algebra. Next, not Euclid, but practical geometry, solid as well as plane; not demonstration, but to make acquaintance. Then not Euclid, but elementary vectors, conjoined with algebra, and applied to geometry ... Elementary calculus should go on simultaneously ... . Euclid might be an extra course for learned men, like Homer. But Euclid for children is barbarous." On the other hand, about 5

years ago, Peter Rudman, a contemporary physicist, wrote this: "High school mathematics education today, ... , all too often neglects the derivations where mathematics is learned and emphasizes memorizing the equations that provide quick solutions in the standardized tests but that are then rapidly forgotten ... ." What aspects of mathematics does each of these authors seem to value most highly and think students should learn? How does what each of them says relate to the ideas of G.H.Hardy in *A Mathematician's Apology*? Why might Heaviside say that teaching Euclid to children is "barbarous?" Was your high school mathematics more or less like what Heaviside is recommending? Was your experience like that Rudman describes? Do you think that emphasizing proofs more would have made mathematics more interesting for more people? Or is that too much to hope for?

2) Alex Bellos, in his book *Here's Looking at Euclid* (catchy title, no?) says, refering to Euclid's proof of Proposition 47 in Book I of the *Elements*, "the thrill of math[ematic]s is the moment of instant revelation, from proofs such as this, when suddenly everything makes sense. It is immensely satisfying, an almost physical pleasure. The Indian mathematician Bhaskara (1114-1185, CE) was so taken by this proof that underneath a picture of it in his twelfth-century math[ematic]s book *Lilavati*, he wrote no explanation, just the word 'Behold!' " If a proof is a journey, can a journey take place instantaneously? How does this relate to some of what Hardy says about the esthetics of mathematics in *A Mathematician's Apology*? Can a picture supply a proof of a mathematical theorem by itself, with no explanation? Is there room for this sort of sudden insight in Lakatos's conception of the development of mathematical understanding through proofs and refutations? Have you ever had this sort of "aha moment" working to find or understand a proof? If so, what had you done before that moment? Was it really instantaneous? And is that flash of insight the only way understanding of mathematics comes to people?

- Option B: Write an essay about the poem *Ithaka* by C. Cavafy posted on our course homepage (I'll provide the text of the poem for you to refer to on the final if you choose this option). In particular, this poem clearly draws on themes from the *Odyssey*, but does it just retell parts of Homer's story, or does it end up making something quite different of them? In particular, is the *return* of Odysseus the main point here? Why doesn't Cavafy mention Telemachus or Penelope? Finally, how do you think what Cavafy is saying here relates to the CHQ theme (especially the "how then shall we live?" part)?