Historia Mathematica **28** (2001), 167–206
doi:10.1006/hmat.2001.2317, available online at http://www.idealibrary.com on IDEAL®

Neither Sherlock Holmes nor Babylon: A Reassessment of Plimpton 322

Eleanor Robson

All Souls College, Oxford OX1 4AL, United Kingdom

E-mail: eleanor.robson@all-souls.ox.ac.uk

Ancient mathematical texts and artefacts, if we are to understand them fully, must be viewed in the light of their mathematico-historical context, and not treated as artificial, self-contained creations in the style of detective stories. I take as a dramatic case study the famous cuneiform tablet Plimpton 322. I show that the popular view of it as some sort of trigonometric table cannot be correct, given what is now known of the concept of angle in the Old Babylonian period. Neither is the equally widespread theory of generating functions likely to be correct. I provide supporting evidence in a strong theoretical framework for an alternative interpretation, first published half a century ago in a different guise. I recast it using regular reciprocal pairs, Høyrup's analysis of contemporaneous "naïve geometry," and a new reading of the table's headings. In contextualising Plimpton 322 (and perhaps thereby knocking it off its pedestal), I argue that cuneiform culture produced many dozens, if not hundreds, of other mathematical texts which are equally worthy of the modern mathematical community's attention.

Wir müssen frühe mathematische Texte und Objekte im Hinblick auf ihre mathematisch-historische Umgebung betrachten und sie nicht als künstliche, vollständige Schöpfungen im Stile von Detektivgeschichten behandeln, wollen wir sie verstehen. Als dramatische Fallstudies dient mir die Keilschrifttafel Plimpton 322. Ich zeige auf, dass die weitverbreitete Ansicht, so etwas wie eine trigonometrische Tabelle vor uns zu haben, nicht richtig sein kann, und zwar aufgrund unseres Wissens über die Vorstellung des Winkels in altbabylonischer Zeit. In gleiche Weise ist die gängige Theorie über erzeugende Funktionen wahrscheinlich falsch. Ich kann meine Neuinterpretation, die in einen stark theoretischen Rahmen eingebettet wird, mit Texten belegen. Hinter meiner Neuinterpretation liegt eine fünfzigjährige Theorie, die auf Bruins zurückgeht. Sie fundiert auf den Gebrauch von regelmassigen, reziproken Paaren, auf Høyrups Analyse der naiven Geometrie und auf eine neue Lesung der Überschriften der Tabelle. Indem ich die Keilschrifttafel Plimpton 322 in ihren historischen Kontext stelle, plädiere ich dafür, dass viele andere mathematische Texte mesopotamischen Ursprungs es ebenso verdienen, von uns beachtet zu werden.

AMS Subject Classifications: 01A17; 01A85.
Key Words: Old Babylonian mathematics; Pythagoras' theorem; reciprocals; Plimpton 322.

NEITHER SHERLOCK HOLMES...

Some years ago, R. Creighton Buck published an analysis of the famous cuneiform tablet Plimpton 322, in an article which he called "Sherlock Holmes in Babylon" [Buck 1980]. His is by no means the only study of the tablet, and what follows is most emphatically not an attack on the work of Buck in particular, which is in many ways considered and sensible, but a refutation of the many dozens of studies that Plimpton 322 has inspired since its publication in 1945. Buck's title articulates most eloquently and engagingly, albeit unwittingly, a common attitude of mathematicians to the history of ancient mathematics over the past half century (and arguably longer): that it can be treated rather like a piece of

0315-0860/01$35.00

detective fiction. The main protagonist is the scholar-sleuth (and note that in most classic detective fiction, our hero(ine) is most decidedly not a professional but a genteel amateur, who outwits the plodding police officers/historians every time). His task is to solve the mystery of the historical document at hand (The Mystery of the Cuneiform Tablet in our case), with a finite, self-selected, set of clues to help him (The Strange Affair of the Numerical Errors). And of course, the setting is a bounded one: not even the isolated country house or the railway carriage, but just the text itself. The real world does not intrude on our scholar-sleuth: historical and linguistic context is an irrelevance, we are implicitly led to believe, which only the dullard history-police choose to bother themselves with. Like the scenarios of detective fiction, pieces of mathematics are self-contained worlds, whose mysteries can be solved by close analysis of nothing but themselves.

But, although it may be argued that the (re)construction of history is nothing more than inventing more or less plausible stories about the past, each of which will differ according to the historians who tell them, the mathematical artefacts of the past most certainly do not themselves resemble the self-contained settings of a country house mystery. Mathematics is, and always has been, written by real people, within particular mathematical cultures which are themselves the products of the society in which those writers of mathematics live. It is the aim of this article to show how dramatically more convincing a story one can tell about Plimpton 322 if it is put into its mathematico-historical setting.

One of the enduring attractions of Plimpton 322 for the mathematical community has been that it exhibits sophisticated and systematic mathematical techniques for an apparently "pure" end, either "number-theoretical" or "trigonometric."[1] But a mathematical culture comprises more than its most spectacular discoveries. It is both pernicious and simple-minded to cherry-pick the "cleverest" or "most sophisticated" mathematical procedures (of any society) to present as the history of mathematics. David Pingree has argued that:

> the [...] attitude that what is valuable in the past is what we have in the present [...] makes historians become treasure hunters seeking pearls in the dung heap without any concern for where the oysters live and how they manufacture gems. [Pingree 1992, 562]

This is exactly what the purveyors of the wonders of Plimpton 322 have, by and large, been doing hitherto. They have also unwittingly perpetuated the colonisation, appropriation, and domestication of the pre-Islamic Middle East by the Western present, as described by Zainab Bahrani:

> It is at once the earliest phase of a universal history of mankind in which man makes the giant step from savagery to civilisation, and it is an example of the unchanging nature of Oriental cultures. [In the Orientalist view] the Mesopotamian past is the place of world culture's first infantile steps: first writing, laws, architecture and all the other firsts that are quoted in every student handbook and in all the popular accounts of Mesopotamia. [Bahrani 1998, 162]

For many people, the attraction of Plimpton 322 has been exactly its status as a "first infantile step" on the way to modern Western-style mathematics. Cooke [1997] for instance, counts Mesopotamia as producing "Early Western mathematics" simply because it predates Classical sources, while consigning Islamic mathematics from the same region to "other traditions" even though the latter arguably had more influence on the West than the former.

[1] As hinted at already by Neugebauer and Sachs [1945, 39].

Further, and as a direct result of this tendency to modernise and domesticate, little distinction has been made between Plimpton 322 the historical artefact and Plimpton 322 the mathematical text. The numerical table purporting to represent the original found in the general history books is a modern text: it is in decimal notation, in modern Indo-Arabic numerals with all errors eradicated, and printed with ink on paper. Moreover, the headings to the columns—the words, in short—may be silently omitted or replaced with (modern) symbolic notation for the variables each heading is supposed to represent. It is easy to forget the chronological distance and the cultural strangeness of the clay tablet itself and to analyse and interpret the paper version as modern mathematics.[2] One might compare reader-centred theories of literary criticism, in which authorial intention is deemed secondary to the meaning extracted from the text by the reader. But the result is mathematics (or perhaps mathematical criticism), but not history.[3] That is not to say that mathematical criticism is not a valid exercise in its own right, but any historical interpretation that it produces may tend to be facile.[4] One characteristic of ancient mathematics that makes it difficult to handle is that it has (almost) no identifiable authors—unlike, say, post-Renaissance European mathematics—and so we cannot begin to treat authorial intent or character. Piotr Michalowski's description of the problems faced by modern readers of anonymous Mesopotamian literature applies equally well to ancient mathematics, I think. For "poet" or "writer" read "mathematician":

> Contemporary readers take for granted certain concepts of authorship and the authority of the writer. Most of these ideas are of fairly recent origin and are very much tied to Western European ideas about identity, originality, individuality, and the high social role of verbal art. It is hardly surprising that most of us find it extremely difficult to shed basic post-Romantic ideas about the spiritual inspirations of writers and the unique talents of poets. [Michalowski 1995, 183]

He goes on to discuss a prominent critic of Victorian literature who has recently turned his attention to the ancient Near East, but who

> refuses to acknowledge the profound differences that separate relatively recent Western ideas about literature and its privileged creators from the conceptions of a culture that prized anonymous composition and in which the caesura between composition and redaction may not have existed, or may have had a very different profile, and in which originality may have been seen in a very different light. [Michalowski 1995, 183]

Replace "literature" with "mathematics," and the analysis remains a useful one.

[2] Schmidt [1980] does exactly this: see pages 20–21. Cf. Knuth [1972, 676] who described a Late Babylonian table of 6-place reciprocals as consisting of a complete, ordered listing of all 231 six-place or shorter regular reciprocal pairs between 1 and 2. In a short note a few years later he admitted he had mistaken Neugebauer's reconstruction for the ancient original (which would have contained only 136 of the 231 pairs, even when complete) [Knuth 1976].

[3] Two very interesting and useful discussions of the applicability of modern literary theory to Sumerian and Akkadian literatures are Michalowski [1995] and Black [1998, 42–48]. They contain much to provoke and stimulate the theoretically minded historian and translator of mathematics, as well as a good deal of useful cultural background on OB mathematics.

[4] One recalls the "rationalist reconstruction" school of the history of the exact sciences, which was entirely internalist in approach. Its followers contended that ancient mathematics could be analysed and lacunae restored solely on the basis of how mathematics happened to look in the mid-20th century, and that they as mathematicians had a privileged reading of that mathematics. The mathematics alone was their subject: its cultural and linguistic setting was considered largely irrelevant [cf. Høyrup 1996, 13–17; Kragh 1987, 161]. The phrase "rationalist reconstruction" was particularly invidious, implying as it did that other approaches were somehow irrational and therefore unscholarly, and that this approach was simply restoring what must have been.

But even when we know nothing about the authors of ancient mathematics as individuals—their names, ages, or even which city or century they lived in—we can now (re)construct a reasonably convincing picture of the cultural milieu in which mathematics was created. The better our understanding of a society, the greater our chances of producing a historically credible analysis of one of its products. When we read Victorian mathematics, for instance, we need little conscious effort to contextualise it because it is chronologically, linguistically, and culturally so close to us. But the more ancient, the more foreign, the more alien the mathematics the greater the need to deliberately set out to explore its setting and to be conscious of the cultural baggage we carry with us as modern readers—and the greater are the inherent difficulties in doing so. For the purposes of the historian there can be no such thing as a Platonic body of mathematics existing independent of human discovery: recorded mathematics is essentially a social product, conceived and articulated by individuals and societies with particular preconceptions, motivations, and modes of communication.

Close reading of the language and terminology of ancient texts is therefore crucial to understanding the conceptual realities that underlie them. Indeed, the interpretation of Plimpton 322 supported here is based on linguistic analysis at two points: Jens Høyrup's understanding of the first, damaged word of the first column, and my analysis of the terms of geometrical shapes. This latter allows the "trigonometric" theory of Plimpton 322 (which has always felt uncomfortable to those of historical sensitivity) to be refuted with more force than before.

In what follows, I first present as neutral a description as I can of the cuneiform tablet, its contents, and a little of the culture which produced it. I then go on to present a composite picture of the generally accepted interpretation of it over the past half century. In the third section, I show why this composite picture cannot be correct, given what is known of the culture which produced Plimpton 322. Finally, I present an alternative interpretation, first put forward shortly after the tablet's publication but largely ignored, which will show Plimpton 322 to be fully integrated into the mainstream of Old Babylonian mathematical thought. My argument does not rest on Holmesian internal mathematical inferences alone, but instead draws on cultural, linguistic, and archaeological evidence too.

. . . NOR BABYLON

Plimpton 322 is the modern label given to a cuneiform tablet written in the ancient Iraqi city of Larsa in the mid-18th century BCE.

Old Babylonian (OB) mathematics, the oldest known body of "pure" mathematics in the world, derived from two separate traditions in early Mesopotamia: an orally-based "surveyors' algebra" and a bureaucratic accountancy culture. The "surveyors' algebra" is heavily based on riddles concerning cut-and-paste geometry and has its origins outside the literate scribal tradition in the late third millennium [cf. Høyrup 1990, 79]. Scribes, on the other hand, had been directly concerned with the quantitative control of goods, time, and labour since the advent of writing at the end of the fourth millennium [Nissen *et al.* 1993; cf. Robson 2000b; 2000c]. Their complex system of metrology, work norms, and other technical constants also reached its apex at the end of the third millennium, under the so-called Third Dynasty of Ur III [Englund 1991; Robson 1999, 138–166]. The two traditions coalesced into the mathematics of the OB humanist scribal schools of the early second millennium, where education appears to have comprised far more than the acquisition of professionally

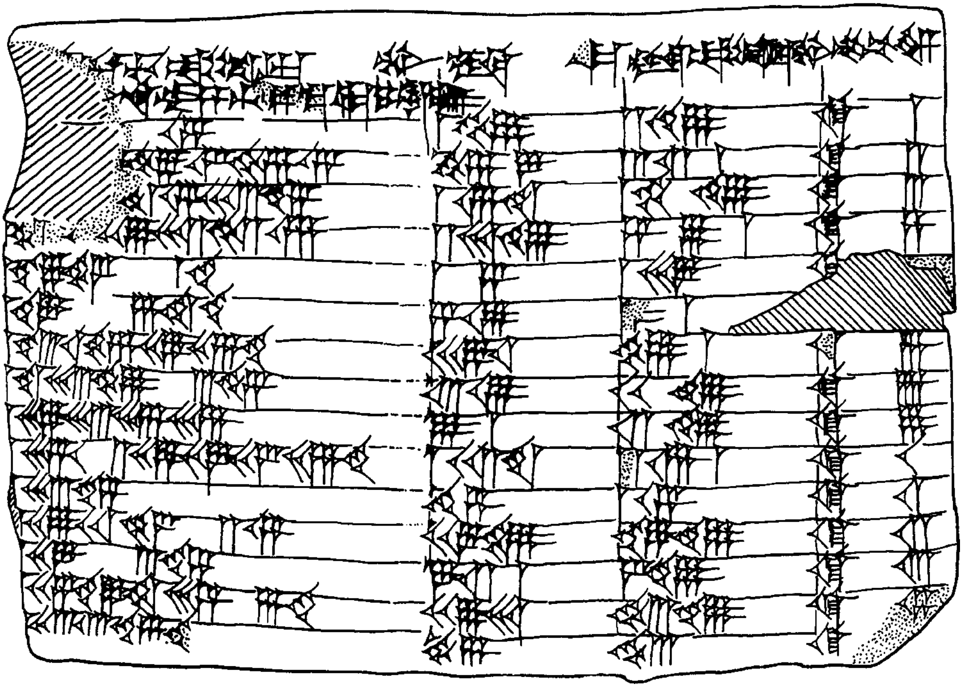

FIG. 1. The obverse of Plimpton 322 (drawing by the author).

useful skills. Although the archaeology of Old Babylonian schools is not clear-cut and the large majority of OB mathematical tablets known are completely unprovenanced, I am convinced that virtually all of the OB mathematical corpus as we have it should be interpreted as the products of scribal training, or, at the very least, as the products of a scholastic milieu. Elsewhere I have elaborated a functional typology of OB mathematical tablets, which shows them to have been written by scribal teachers or by students learning mathematics by rote or by practice [Robson 1999, 174–183]. I certainly do not feel justified in referring to the authors and copyists of OB mathematics as "mathematicians," with the connotations of creativity and professionalism this word carries; I prefer the more neutral "scribes." On the other hand, I have no hesitation in using the pronoun "he" to describe them, OB scribal culture having been almost exclusively male.

Plimpton 322 is, physically at least, a typical product of OB mathematical culture (Fig. 1). It is a clay tablet, measuring some 12.7 × 8.8 cm as it is preserved, ruled into four columns. It was excavated illegally sometime during the 1920s, along with many thousands of other cuneiform tablets, not from Babylon but from the ancient city of Larsa (modern Tell Senkereh, 31°14′ N, 45°51′ E [Roaf 1990, 231]).[5] Its provenance is

[5] If Sherlock Holmes had been in Babylon he would have been wasting his time: no OB mathematics is known to have come from that site—some 200 km north of Larsa—whose lower archaeological strata are virtually inaccessible due to the high water table and Iraqi renovations of the Neo-Babylonian city in the 1980s [Kuhrt 1995, 108–109].

given as “Senkereh” on an undated price list now housed with the Plimpton Collection at Columbia [Banks n.d.]; this fits the characteristics of the headings, based on palaeographic and orthographic comparison with non-mathematical tablets from Larsa now in the Ashmolean Museum, Oxford. A lot of Larsa material was reaching western collections (e.g., the Ashmolean, the Louvre, and the Yale Babylonian Collection) from the antiquities market in the 1920s, at the same time that Plimpton was buying tablets [Banks & Plimpton 1922–1923, 1923]. Tabular formatting was first used in institutional bureaucracies at nearby Nippur from around 1850 BCE, while the earliest attested administrative tables from the kingdom of Larsa date to the period 1837–1784 BCE. Tables which, like Plimpton 322, are on “landscape” oriented tablets with calculations running horizontally across the tablet, and whose final heading is MU.BI.IM “its name” (of which more below), are attested from 1822 BCE onwards [Robson 2000d]. This allows us to confidently date Plimpton 322 to the 60 years or so before the siege and capture of Larsa by Hammurabi of Babylon in 1762 BCE.

The name “Plimpton 322” denotes that it is the 322nd item in Mendelsohn’s catalogue of the cuneiform tablets held by the libraries of the University of Columbia, New York. The entire catalogue entry reads:

> 322. Clay tablet, left-hand edge broken away, bottom of right-hand corner, and a piece of columns 3 and 4 chipped off; fairly well preserved, dark-brown. 8.8 × 12.5 cm.; on obverse 4 columns with 16 lines, reverse blank. Content: Commercial account. No date. [Mendelsohn 1943, 71]

The tablet had originally been acquired by the New York publisher George Arthur Plimpton for his private collection of historical mathematical artefacts, which was bequeathed to Columbia shortly before his death in 1936. He had bought it from the notorious dealer Edgar J. Banks [Banks n.d.], but it is extremely unlikely that either men understood its importance for the history of mathematics [cf. Plimpton 1993; Donoghue 1998].[6] Given the striking similarity in format and lexis between Plimpton 322 and the other early tables from the Larsa mentioned above, it is hardly surprising that its true character passed unnoticed by dealer, owner, and cataloguer alike.

The left-hand side of the tablet is missing, and has been at least since Plimpton acquired it.[7] There is a clean break here, along one of the vertical rulings which divide the surface of the tablet into columns. Traces of glue remain in this break, and it has been implied that the other fraction of the tablet must therefore have been lost in modern times, deliberately or otherwise. However, it was not unusual for unscrupulous early 20th-century antiquities dealers to manufacture “whole” tablets out of disparate fragments in order to attract a higher price for them. Such an occurrence is described, for instance, by Neugebauer and Sachs [1945, 24 n. 88]. In the case of Plimpton 322 it is likely that Banks, who had been a professional Assyriologist, was scrupulous enough to remove the extraneous matter before putting the tablet up for sale. Since it is impossible as matters stand to determine what was originally attached to the tablet as we have it, or where it might be now, we will treat only what is extant, and leave speculative lacunae-filling for later.

[6] I intend to deal with Plimpton and Smith’s acquisition of cuneiform tablets at greater length in the near future.

[7] Compare the catalogue entry just cited with Banks’ description [Banks n.d.]: “A very large burned tablet with one edge broken away, but with the inscription practically complete.” See too the photo in Neugebauer & Sachs [1945, pl. 5] which shows the tablet in exactly the same condition as in Mendelsohn [1943, pl. 2].

TABLE 1
The Extant Contents of Plimpton 322, with Errors Corrected and the Third Element of the Triple Added

I. (damaged), d^2/l^2 or b^2/l^2	II. Square-side of the width, *b*	III. Square-side of the diagonal, *d*	IV. Its name	(Square-side of the length, *l*)
(1) 59 00 15	1 59	2 49	1	2
(1) 56 56 58 14 50 06 15	56 07	1 20 25	2	57 36
(1) 55 07 41 15 33 45	1 16 41	1 50 49	3	1 20
(1) 53 10 29 32 52 16	3 31 49	5 09 01	4	3 45
(1) 48 54 01 40	1 05	1 37	5	1 20
(1) 47 06 41 40	5 19	8 01	6	6
(1) 43 11 56 28 26 40	38 11	59 01	7	45
(1) 41 33 45 14 3 45	13 19	20 49	8	16
(1) 38 33 36 36	8 01	12 49	9	10
(1) 35 10 02 28 27 24 26 40	1 22 41	2 16 01	10	1 54
(1) 33 45	45	1 15	11	1
(1) 29 21 54 2 15	27 59	48 49	12	40
(1) 27 00 03 45	2 41	4 49	13	4
(1) 25 48 51 35 6 40	29 31	53 49	14	45
(1) 23 13 46 40	28	53	15	45

Each of the four remaining columns contains a heading in a mixture of Sumerian and Akkadian and 15 rows of numerical data. These rows almost fill the obverse of the tablet; the reverse is blank, but has been ruled to accommodate the continuation of the columns. The tablet would have been written and read from left to right, but for the moment it is easier to explain its contents moving from right to left. Column IV is headed MU.BI.IM, the Sumerogram for Akkadian *šumšu* "its name," and the numbers below it are simply line numbers, from 1 to 15. Their extra-mathematical function can be seen in their form:[8] whereas mathematical number notation writes the digits to 9 in up to three vertically ranged rows of three vertical wedges, non-mathematical units are written in only two vertically ranged rows, with four as two above two wedges (cf. three above one) and seven to nine as four above three, four above four, and five above four wedges respectively. The middle two extant columns, from right to left still, are headed ÍB.SI$_8$ (=*mitharti*) *ṣiliptim* "the square of the diagonal" and ÍB.SI$_8$ SAG (=*mitharti pūtim*) "the square of the short side." They contain the lengths of the hypotenuses and widths respectively of 15 right-angled triangles (Table 1). Following Neugebauer and Sachs' original notation [1945, 39] we can abbreviate d = diagonal or hypotenuse, l = long side or length, and b = short side or width, while keeping in mind that this notation is for our convenience only and had no part to play in ancient mathematics.

Why do the headings refer to squares if the columns contain lengths of lines? The answer lies in the Akkadian word *mithartum*, a noun derived from the reflexive stem of the verb *mahārum* "to be equal and/or opposite," which literally means "thing which is equal and opposite to itself." In Akkadian, as in other languages, the word for "square" can also refer

[8] *Pace* Friberg [1981a, 295], who sees their form as indicative that the tablet is "a copy of an older original"; he does not convincingly explain, however, why (if this were the case) the orthography on the rest of the tablet is not similarly archaic.

to its side and therefore also to "square root" [Høyrup 1990a, 49–50]. So here *mithartum* (in its construct state before the genitives of *ṣiliptum* and *pūtum* indicating possession) must mean not "square" but "square-side" or perhaps "square root." In fact this dual meaning for *mithartum* is not as confusing as it first appears. The OB scribes used the sexagesimal place value system of abstract number solely for calculating: the dimensions, weights, and capacities of the subjects of their calculation were invariably recorded in the most suitable metrological units. With two separate systems for length and area it would have been almost impossible for a scribe to confuse the side of a square with its area—which in any case was almost invariably written explicitly as A.ŠÀ *mithartim* "the area of a square." For instance, when we read 1 UŠ *mithartum* in the first line of every problem in BM 15285 (probably from Larsa; [Robson 1999, 208–217]) we have to translate "1 UŠ is the square-side" and not "1 UŠ is the square-area" because the UŠ is a unit of the linear metrological system (ca. 360 m), not of the areal system. There is no ambiguity. Difficulties of interpretation arise—for us—only when we are dealing with decontextualised calculations in the sexagesimal place value system, which were never intended by the scribes to be visible in the final text.[9]

The heading of the left-hand column is somewhat damaged (we will return to what it might say later), but the 15 rows beneath it contain *either* the ratio of the square on the diagonal to the square on the long side *or* the ratio of the square on the short side to the square on the long side. This ambiguity arises from the nature of the break described earlier: the tablet cracked along the left edge of the column, and the impressions that remain may simply be intersections of the horizontal row rulings and the vertical column line (in which case the second reading is correct) or they are the remains of a series of vertical wedges representing 1s, with which each number in the column originally began (in which case we have to accept the first interpretation). We leave the matter unresolved for the moment but remark that, either way, the column is sorted in descending numerical order (see Table 1).

It is well known that there are seven errors in the tablet (see Table 2), some of which are clearly trivial copying errors, but three have a bearing on how the table should be interpreted. We will come back to these later.

The sexagesimal place-value system does not mark absolute value—there is no mechanism for showing zeros of any kind, or the boundary between integers and fractions—which makes any decimal translation for Table 1 to some extent an arbitrary one. In Table 3 I have chosen, following conventional practice, to assume that the values in Columns II–III are integers.

AN ANALYTICAL FRAMEWORK: SIX CRITERIA FOR ASSESSMENT

Plimpton 322 has undoubtedly had the most extensive publication history of any cuneiform tablet, mathematical or otherwise. After its publication in *Mathematical Cuneiform Texts* [Neugebauer & Sachs 1945, Text A], Neugebauer discussed the tablet further in his highly influential *Exact Sciences in Antiquity* [Neugebauer 1951]. This book was almost certainly its primary source of dissemination into the wider mathematical community and all the

[9] This duality in the meaning of "square" can be traced right back to the mid-third millennium table of squares VAT 12593 from Shuruppag, in which the sides, in descending order, are written in linear measure followed by the sign sá "equal" (cf. the OB Sumerogram ÍB.SI$_8$ 'square,' where SI$_8$ = sá), and their products are given in area measure [Nissen *et al.* 1993, 136–139].

TABLE 2
The Errors in Plimpton 322

Column	Line	Error	Comments
I	2	56 for 50 06	Simple copying error: two places conflated
	8	59 for 45 14	Simple arithmetical error: two places added together
	13	27 03 45 for 27 00 03 45	Simple copying error: the empty sexagesimal place should be marked with a blank space
II	9	9 01 for 8 01	Simple copying error: 9 looks very similar to 8 in cuneiform
	13	7 12 01 for 2 41	Calculation error: square of the correct value
II or III	15	56 for 28 or 53 for 1 46	Calculation error: twice or half the correct value[a]
III	2	3 12 01 for 1 20 25	Calculation error: no obvious numerical relationship

[a] It has more commonly been proposed that the error lies in the third column (half the correct value) and that the correct triple is 56, 1 30, 1 46 [e.g., Neugebauer & Sachs 1945, 38, 40–41], but it is also possible that the second column contains the erroneous value and the triple should be 28, 45, 53 [e.g., Friberg 1981a, 288]. This is my favoured interpretation, discussed further below, and is used throughout this article, except where necessary in Table 4.

standard mathematical history books (where it is still found today). The most important post-*ESA* studies were by Bruins [1955], Price [1964], Buck [1980], Schmidt [1980], and Friberg [1981a]. Buck also refers to the theories of a certain Voils, whose work I have not managed to trace.

Before examining the relative merits of these works, we need to be clear about the criteria we should use to judge them. Moving from the external toward the internal and

TABLE 3
Decimal Translation of Plimpton 322

I. (damaged), d^2/l^2 or b^2/l^2	II. Square-side of the width, b	III. Square-side of the diagonal, d	IV. Its name	(Square-side of the length, l)
(1).9834028	119	169	1	120
(1).9491586	3367	4825	2	3456
(1).9188021	4601	6649	3	4800
(1).8862479	12,709	18,541	4	13,500
(1).8150077	65	97	5	72
(1).7851929	319	481	6	360
(1).7199837	2291	3541	7	2700
(1).6845877	799	1249	8	960
(1).6426694	481	769	9	600
(1).5861226	4961	8161	10	6480
(1).5625	45	75	11	60
(1).4894168	1679	2929	12	2400
(1).4500174	161	289	13	240
(1).4302388	1771	3229	14	2700
(1).3871605	28	53	15	45

from the general towards the specific, a satisfactory theory about Plimpton ought to fulfil these conditions:

1. *Historical sensitivity*. The theory should respect the historical context of Plimpton 322 and not impose conceptually anachronistic interpretations on it.
2. *Cultural consistency*. The theory should explain the workings and function of Plimpton 322 in the light of what is known of the rest of OB mathematics, ideally using evidence from Larsa and its environs.
3. *Calculational plausibility*. The theory should show how Plimpton 322 was calculated (and the errors miscalculated) with arithmetical techniques known from other OB tablets, preferably from Larsa.
4. *Physical reality*. When making restorations to the left of the extant columns, the theory should acknowledge that Plimpton 322 is an archaeological artefact and not a disembodied text. The restoration should result in a (virtual) cuneiform tablet of plausible size and proportions.
5. *Textual completeness*. The theory should account for, not only the numerical contents of the tablet, but also its four column headings, explaining how they relate to the numbers below and supplying a linguistically and mathematically plausible reconstruction of the damaged words in Column I. Every word should count for as much as every number.
6. *Tabular order*. The theory should explain the logical order of the columns, from left to right, and of the rows, from top to bottom, including any proposed restorations to the left of the tablet or on the reverse. In other words, it should not violate the "grammar" of OB mathematical tables. By this I mean that all numerical tables, whether their contents are mathematical, economic, astronomical, or otherwise, are ordered from top to bottom, and from left to right, and sorted in ascending or descending numerical order by the contents of the first, i.e., leftmost column. It is well known (but nowhere explicitly stated, I think) that multiplication and metrological tables, as well as associated tables such as square roots, behave thus [e.g., Neugebauer & Sachs 1945, 11–36], but all known tabular calculations do too: nine unprovenanced [Neugebauer & Sachs 1945, 17–18]; five from Ur [Robson 1999, 264–266]; one from Nippur [Robson 2000a, no. 10]. In short, Column I should not be understood as derivative of Columns II or III but rather as a step on the way to calculating first II and then III. It should not be assumed that there is anything missing to the right of the tablet (such as the lengths of the long sides). Similarly, the theory should make restorations to the left of the extant columns which are in logical columnar order from left to right and the contents of whose leftmost column are in ascending or descending numerical order.

Armed with these six desiderata, we are now in a strong position to gauge how plausible the various schools of thought about Plimpton 322 really are.

THE RECEIVED WISDOM

The standard theory to account for the construction of the table assumes that it was generated by a set of pairs p, q (or in some notations u, v). This Aaboe summarises in a particularly inappropriate modernising "theorem":

TABLE 4
So-Called Generating Numbers for Plimpton 322

p	q	p^2	q^2	$2pq$	p^2-q^2 (Col. II)	p^2+q^2 (Col. III)	(Col. IV)	p/q
12	5	2 24	25	2 00	1 59	2 49	1	2;24
1 04	27	1 08 16	12 09	57 36	56 07	1 20 25	2	2;22 13 20
1 15	32	1 33 45	17 04	1 20 00	1 16 41	1 50 49	3	2;20 37 30
2 05	54	4 20 25	48 36	3 45 00	3 31 49	5 09 01	4	2;18 53 20
9	4	1 21	16	1 12	1 05	1 37	5	2;15
20	9	6 40	1 21	6 00	5 19	8 01	6	2;13 20
54	25	48 36	10 25	45 00	38 11	59 01	7	2;09 36
32	15	17 04	3 45	16 00	13 19	20 49	8	2;08
25	12	10 25	2 24	10 00	8 01	12 49	9	2;05
1 21	40	1 49 21	26 40	1 48 00	1 22 41	2 16 01	10	2;01 30
2	1	4	1	4	3	5	11	2
48	25	38 24	10 25	40 00	27 59	48 49	12	1;55 12
15	8	3 45	1 04	4 00	2 41	4 49	13	1;52 30
50	27	41 10	12 09	45 00	29 31	53 49	14	1;51 06 40
9	5	1 21	25	1 30	56	1 46[a]	15	1;48

[a] The values of length ($2pq$), width ($p^2 - q^2$) and diagonal ($p^2 + q^2$) in this row are twice those expected by the method preferred here; see note to Table 2.

If p and q take on all whole values subject only to the conditions

1) $p > q > 0$,
2) p and q have no common divisor (save 1),
3) p and q are not both odd,[10]

then the expressions

$x = p^2 - q^2$, [our b]
$y = 2pq$, [our l]
$z = p^2 + q^2$, [our d]

will produce all reduced Pythagorean number triples, and each triple only once. [Aaboe 1964, 30–31]

Whether or not this theorem, or variants on it, accounts for exactly all the triples in the table (see Table 4), there are several strong arguments against Plimpton 322 having been generated directly from p and q.

First, Neugebauer and Sachs noted that the values of p and q were all to be found in the standard reciprocal tables (Table 5), except 2 05 (row 4), which was the normal starting point for an extension of those tables. With 44 different numbers in the table, the scribe would have $44 \times 43/2 = 946$ pairs to choose from. Now, supposing the scribe was familiar with the idea of odd and even (for which concept there is no Old Babylonian evidence), he could eliminate a further $12 \times 11/2 = 66$ possible pairs of odd numbers from his choices, leaving 880 available starting points. And if we suspend our historical judgement even further for

[10] But note that the last line of the table does not fulfil this condition, unless we take the reduced set (28, 45, 53) and make $l' = 28$, $b' = 45$, instead of $l' = 45$, $b' = 28$ (and p/q =3;30, clearly outside the decreasing sequence). Much effort has been expended by several scholars on determining the restrictions the ancient scribe “must” have placed on p and q.

TABLE 5
The Standard Table of Reciprocals[a]

2	30	8	7 30	16	3 45	27	2 13 20	45	1 20	1 04	56 15
3	20	9	6 40	18	3 20	30	2	48	1 15	1 12	50
4	15	10	6	20	3	32	1 52 30	50	1 12	1 15	48
5	12	12	5	24	2 30	36	1 40	54	1 06 40	1 20	45
6	10	15	4	25	2 24	40	1 30	1	1	1 21	44 26 40

[a] This consists of reciprocal pairs for all one-place regular numbers, plus the pairs for the squares of 8 and 9 (the squares of 1–6 and 10 are already in the list). Optionally, the table may also include the three pairs between 64 and 81 but these are all reversed repetitions of pairs which occur earlier in the table.

a moment and imagine that the scribe was familiar with the idea of coprimes, he would be left with a mere 159 admissible pairs. How would the scribe have known which fifteen to choose? Or are we to assume that he worked through them all? It has been pointed out that the ratio of p to q descends steadily through the table from 2;24 to 1;48 (see the last column of Table 4), but we are still left wanting an explanation as to how those ratios were found and sorted.

Second, and perhaps more seriously, any convincing contender for an explanation of Plimpton 322 must take the order of columns into account (Criterion 6, above). Column I of Plimpton 322 as extant (containing values of $[(p^2+q^2)/2pq]^2$ under this scheme) is quite extraneous to the calculation scheme proposed. Because it occurs to the left of Columns II and III (p^2-q^2 and p^2+q^2 respectively) we should expect it, according to the usual grammar of Old Babylonian mathematical tables, to contain calculations intermediate to the results in II or III or both. Plausible candidates might be p^2 or q^2, for instance. But instead we have results apparently derived from Column III.[11] Neither can the heading of Column I be made to fit the p, q interpretation. Even Neugebauer and Sachs' hesitant attempt at translation came up with "The *takiltum* [untranslatable] of the diagonal which has been subtracted such that the width . . . ." They admitted, "We are, however, unable to give a sensible translation of this passage leading to $d^2/(d^2-b^2)$" [Neugebauer & Sachs 1945, 40]. According to Criterion 6, the ordering of Column I must arise as a consequence of the numerical ordering of the column(s) preceding it—but clearly neither of the p and q columns of Table 4 (which are supposedly the missing columns of the tablet) is in a sorted numerical order.

Third, what of the errors in the table (cf. Table 2)? Discounting the four mistakes which probably occurred in the transfer of data from rough calculating tablet to the clean copy which is Plimpton 322,[12] we have to explain how they might have arisen. Much has been said by others on the three nontrivial errors but the crux of the argument, according to its proponents, lies in the explanation of line III 2, which has no simple numerical relationship to the correct value (p^2+q^2 in the p, q theory):

[11] On similar grounds we cannot assume that the missing columns contained the long sides—unless we can show that it was a starting point, by-product, or step along the way to calculating the two extant elements.

[12] Ur III and OB scribes were trained to destroy their rough calculations, either by overwriting or by erasing the tablet on which they were written [Powell 1976, 421; Robson 1999, 10].

> It seems to me that this error should be explicable as a direct consequence of the formation of the numbers of the text. This should be the final test for any hypothesis advanced to explain the underlying theory. [Neugebauer 1951, 50 n. 20]

Gillings [1953, 56] and Neugebauer [1951, 50 n. 20] agree that the extant value arose from a cumulative series of errors. First, for reasons unexplained by Gillings or Neugebauer, the scribe chose to calculate $p^2 + q^2$ not as a simple sum of two squares ($1\ 04^2 + 27^2$) but as $(p + q)^2 - 2pq$. However, in finding the second element he failed to multiply by p, stopping at $2q$ (or alternatively taking $p = 1\ 00$, not 1 04). Then, instead of subtracting the resultant 54 from $(1\ 04 + 27)^2 = 2\ 18\ 01$ as required, he added it to arrive at 3 12 01. In short, we are asked to believe that, as well as making an unremarkable arithmetical error, the scribe first failed to use the values of p^2, q^2, and $2pq$ which he must already have found in determining Columns I and II, either here alone or throughout the whole tablet, in the most direct method available; and then made a mess of the more complicated alternative (but solely in this line), adding where he should have subtracted.

Price [1964, 9], on the other hand, believes that the scribe did take the direct $p^2 + q^2$ route, but having safely calculated that $1\ 04^2 = 1\ 08\ 16$ he must have incorrectly taken $27^2 = 2\ 03\ 45 = 27 \times 25 \times 11$. Why or how the scribe should have done that, Price cannot say.

In short, the p, q theory has these weaknesses: it is over-reliant on mathematical knowledge for which there is at best dubious corroborating historical evidence; it grievously contravenes the strong conventions of OB table-making by failing to provide an ordered list at the beginning of the table; it fails to explain the Column I heading, the position of Column I in the table, or even its occurrence at all;[13] and fails to account simply and convincingly for the nontrivial errors which appear in Plimpton 322. In other words, it fails to satisfy Criteria 1–3 and 5–6 and does not engage with Criterion 4; Criterion 5 is discussed further below. On these grounds we can no longer consider the p, q theory a satisfactory interpretation of Plimpton 322.

RIGHT ANGLES AND WRONG ANGLES: CIRCLE MEASUREMENT IN THE OLD BABYLONIAN PERIOD

So much, for the moment, for how the table might have been composed. What of its supposed function? One of the most popular theories (tending to crystallise in unpublished manuscripts and short discussions in the general histories [e.g., Joyce 1995; Calinger 1999, 35–36]) is that Plimpton 322 represents a trigonometric table of some type. This interpretation—which should make anyone with a passing knowledge of the post-Ptolemaic development of trigonometry feel more than a little uncomfortable—appears to be the bastard offspring of a passing remark made by Neugebauer and Sachs:

> Formulating the problem with respect to the triangles, we can say that we start out with almost half a square (because the value of $b : l$ which corresponds to the first line is 0;59,30) and gradually diminish the angle between l and d step by step, the lowest value being almost exactly 31°. [Neugebauer & Sachs 1945, 39]

[13] See already Buck [1980, 343].

Nowhere, however, in their concluding discussion of "historical consequences" did they mention "trigonometry" or "angle," but kept their comments to the (equally dubious) "purely number theoretical character" of Plimpton 322 [Neugebauer & Sachs 1945, 41]. Neugebauer reworded these comments slightly later on but he still refrained from concluding that the table was in any way trigonometric in character:

> If we take the ratio b/l for the first line we find 1,59/2,00 = 0;59,30 that is, almost 1. Hence the first right triangle is very close to half a square. Similarly, one finds that the last right triangle has angles close to 30° and 60°. The monotonic decrease of the numbers in Column I suggests furthermore that the shape of these triangles varies rather regularly between those two limits. [· · ·] This observation suggests that the ancient mathematician was interested not only in determining triples of Pythagorean numbers but also in their ratios d/l. [Neugebauer 1951, 38]

Elsewhere I have argued that all Old Babylonian area-geometry is based on defining components: external lines (which may be straight or curved) from which the area of that figure is defined and calculated [Robson 1999, 47–60]. In many cases, the names for the defining component and the figure itself—by which I do not mean the *area* of the figure—are identical. Both the circle and the circumference are called in Old Babylonian *kippatum* from the verb *kapāpum* "to curve." Both the square and its side are *mithartum* from the reflexive stem of *mahārum* "to be equal and opposite"—as we have already seen—and the rectangle and its diagonal are *ṣiliptum* from *ṣalāpum* "to strike through" (but of course, in this last case, a diagonal does not uniquely define its surrounding rectangle; it only defines the configuration as the [simplest] figure possessing a diagonal, i.e., a rectangle). The conceptual identity of the (two-dimensional) circle and its (one-dimensional) circumference is revealed not just in the terminology, however, but also in the means by which circles were dealt with geometrically.

Two nice examples of OB circles can be seen on two clay tablets now owned by the University of Yale (YBC 7302 and YBC 11120 [Neugebauer & Sachs 1945, 44]; probably from Larsa). The tablets themselves are circular in shape—a little like large biscuits, around 8 cm in diameter—which suggests strongly that they were written by students, as rough work [Robson 1999, 10–12]. In modern transcription they look as shown in Fig. 2. The numbers on the first circle are easy to read: as they are all less than 60 we can treat them like modern decimals. We can see immediately that $9 = 3^2$ and that $45 = 5 \times 9$. Because 45 is in the middle of the circle, we can guess that this is meant to be its area (which we will call A), and we can guess too that either 3 or 9, on the outside of the circle, is meant to be its circumference (which we will call c). We know that $A = \pi r^2$, but we have no

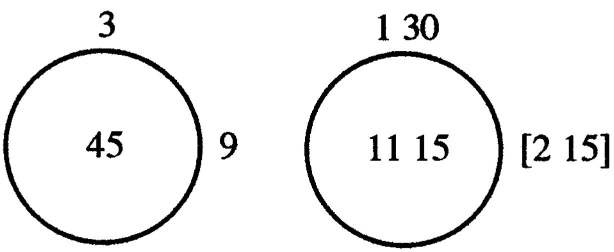

FIG. 2. Two Old Babylonian circles, YBC 7302 and YBC 11120 (after Neugebauer & Sachs [1945, 44], drawing by the author).

radius r marked on the circle. We also know that $c = 2\pi r$, so—by some *modern* algebraic manipulation—we can see that $A = c^2/4\pi$. So it looks as through 3 is the length of the circumference, and that 9 is its square, c^2.

Substituting these values into our formula, we have

$$A = \frac{9}{4\pi} \approx \frac{9}{4 \times 3} = 9 \times 0;05 = 0;45,$$

or 3/4.[14] Checking this formula against the second circle, with c = 1;30, gives

$$A = \frac{1;30^2}{4\pi} \approx \frac{2;15}{4 \times 3} = 2;15 \times 0;05 = 0;11\ 15$$

or $11/60 + 15/3600 = 3/16$ in modern fractions. In other words, a circumference which is half the length of the first surrounds a circle with a quarter of that first area, as we would expect.[15]

Apart from the arithmetical difference of having to work in base 60, these two examples illustrate beautifully a fundamental distinction between the modern circle and the ancient. Whereas we are taught to conceptualise the circle as a figure generated by a radius rotating around the centre (as with a pair of compasses, and our formula $A = \pi r^2$), in the Old Babylonian period it was seen as the figure surrounded by a circumference. There are several other contemporary texts which corroborate this interpretation. For instance, YBC 7997: I 9–II 3 ([Neugebauer & Sachs 1945, text Pa]; probably from Larsa) gives instructions for finding the area of a circle with the same circumference as the second example above:[16]

> 1 30 *ki-*[*ip-pa*]*-tam* / [*tu*]*-uš-ta-ka-al* / *a-na* 5 *tu-ub-ba-al-ma* / 11 15 *i-li-a-ma*
>
> You square 1;30, the circumference. You multiply (the result) by 0;05 and 0;1115 will come up, and (. . . the text continues).

Haddad 104 (iii): II 26-8 ([Al-Rawi & Roaf 1984, 188–195, 214]; from Mê-Turān) shows that the circle's area is calculated from the circumference, even when the diameter is known (in this case the subject is a cylindrical log with longer diameter at the bottom than at the top):

> 1 40 *mu-uh-hi i-ṣí-im šu-ul-li-iš-ma* / 5 *ki-pa-at i-ṣí-im i-lí* / 5 *šu-ta-ki-il-ma* 25 *i-lí* / 25 *a-na* 5 *i-gi-gu-bé-em i-ši-ma* / 2 05 A.ŠÀ *i-lí*
>
> Triple 1;40, the top of the log, and 5, the circumference of the log, will come up. Square 5 and 25 will come up. Multiply 25 by 0;05, the coefficient, and 2;05, the area, will come up.

[14] 3 was a standard approximation to π in Old Babylonian mathematics: although it was known to be inaccurate it was easy to calculate with. In certain circumstances the much more accurate 3;07 30, or 3 1/8 was used. The "coefficient of a circle," 0;05, was listed in most known Mesopotamian coefficient lists [Robson 1999, 34–38].

[15] Cf. AUAM 73.2841 [Sigrist 2001, no. 224], on a damaged square tablet which shows, like YBC 7302, a circle inscribed with the number 45. The number 3 is written twice, in vertical alignment, to the left of the circle, with the number 9 in the bottom left hand corner of the tablet. Although unprovenanced, the tablet is certainly OB.

[16] The mark / shows the division of physical lines on the tablet. Square brackets [ ] mark signs that are completely missing.

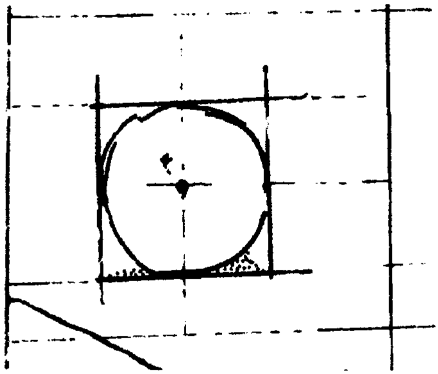

FIG. 3. An Old Babylonian circle drawn with compasses (detail of BM 15285, drawing by the author).

Although the diameter *tallum* regularly crops up in OB problems about circles—and, one could argue, it was necessary in order to conceive of a circumference or circle as a loop whose opposite points are all equidistant—the radius *pirkum* is never mentioned.[17] That is not to say, though, that the radius played no part in OB geometry. We find it, for instance, in problems about semicircles. In those contexts, however, it functions as the short transversal of the figure, perpendicular to the diameter or long transversal, *tallum*. Indeed, this is the function of the *pirkum* in the context of all OB geometrical shapes [Robson 1999, 38]; it is never conceptualised as a rotatable line.

Figure 3, from a "text-book" on finding the areas of geometrical figures, shows a circle inscribed in a square (BM 15285 [Robson 1999, 208–217]; probably from Larsa). It gives clear evidence that circles could be drawn with compasses from a central point. My argument is certainly not that the radius was not known in the Old Babylonian period, but simply that it was not central to the ancient mathematical concept of a circle.

In short, to treat Plimpton 322 as a trigonometric table of any kind does extreme violence to Criteria 1–2. The Old Babylonian circle was a figure—like all OB geometrical figures—conceptualised *from the outside in*. In such a situation, there could be no notion of measurable angle in the Old Babylonian period.[18] Without a well-defined centre or radius there

[17] The sole exception is the sequence of coefficients on *TMS* 3 [Bruins & Rutten 1961, texte 3] from Susa:

D 2 5 IGI.GUB *šà* GÚR	0;05, the coefficient of a circle.
D 3 20 DAL *šà* GÚR	0;20, the diameter of a circle.
D 4 10 [*pi*]-*ir-ku šà* GÚR	0;10, the radius of a circle.

But even here the coefficient of the radius ("short transversal") is given last, after the area and diameter. These three entries in the list start a long section recording the coefficients of the areas, diameters (long transversals), and radii (short transversals) and other components of seven different geometrical figures. In each case the short transversal is considered to be fixed perpendicular to the diameter and measured from the diameter to the perimeter of the figure [Robson 1999, 199–200].

[18] The illustration does show, though, that there was a concept of "quasi-perpendicularity," or *more-or-less* right-angledness. The squares on the tablet are definitely skew: it is not my drawing skills at fault, but the ancient scribe's conception of squareness. We do not know exactly what the Old Babylonian perception of perpendicularity was, but it seems to have been something like the range of angles for which the "Pythagorean Rule" is reasonably accurate [cf. Høyrup 1999a, 403].

could be no mechanism for conceptualising or measuring angles,[19] and therefore the popular interpretation of Plimpton 322 as some sort of trigonometric table becomes meaningless.[20] Any plausible hypothesis about the creation of Plimpton 322 must therefore see the more-or-less linear decrease in the Column I values not as a goal of the text but as an incidental by-product with no quasi-trigonometric significance for its ancient creator. It is more historically sound to view the starting point of the table as a triangle which is more-or-less half a square, and the (unattained) end-point as a triangle with long side and short side in either of the ratios 3 : 2 or 2 : 1.

RECIPROCAL PAIRS AND CONCRETE GEOMETRY: AN ALTERNATIVE INTERPRETATION OF PLIMPTON 322

Jens Høyrup's ground-breaking work over the last decade on the terminology of Old Babylonian "algebra" has revealed an underlying "cut-and-paste" or "naive" geometric methodology, exemplified by the problem on YBC 6967 ([Neugebauer & Sachs 1945, text Ua; Høyrup 1990a, 262–266]; probably from Larsa):[21]

> [IGI.BI] *e-li* IGI 7 *i-ter* / [IGI] *ù* IGI.BI *mi-nu-um* / *at-ta* 7 *ša* IGI.BI / UGU IGI *i-te-ru* / *a-na ši-na he-pé-ma* 3 30 / 3 30 *it-ti* 3 30 / *šu-ta-ki-il-ma*! 12 15 / *a-na* 12 15 *ša i-li-kum* / [1 A.ŠÀ]-*la-am ṣí-ib-ma* 1 12 15 / [ÍB.SI$_8$ 1] 12 15 *mi-nu-um* 8 30 / [8 30 *ù*] 8 30 *me-he-er-šu i-di-ma* / 3 30 *ta-ki-il-tam* / *i-na iš-te-en ú-su-uh* / *a-na iš-te-en ṣi-ib* / *iš-te-en* 12 *ša-nu-um* 5 / 12 IGI.BI 5 *i-gu-um*

[19] The slope of walls (and other structural lines which deviate from the vertical) is described in formulations such as *i-na* 1 KÙŠ $\frac{1}{3}$ KÙŠ KÚ Ì.KÚ-*ma* "in 1 cubit (height) the slope slopes 1/3 cubit" said of a wall (YBC 4673 (xii): IV 12 [Neugebauer 1935–1937, III 29–34; Robson 1999, 90]; from Larsa?) or *i-na* 1 KÙŠ 1 30 KÙŠ ŠÀ.GAL "in 1 cubit (height) the slope is 1;30 cubits" said of a grain-pile (BM 96954 + BM 102366 + SÉ 93 (xv): III 28′–29′ [Robson 1999, 222]; from Sippar).

[20] As Jens Høyrup (personal communication) points out, "the trigonometric claim is as meaningless as a claim that the sequence 1–2–3–4 ··· in itself, and with no corresponding angles listed, constitutes a table of tangents."

[21] See also VAT 8520 ([Neugebauer 1935–1937, I 346–351]; probably from near Larsa) for two further problems on reciprocal pairs, with more complicated starting conditions.

The possible attribution of these two tablets to the Larsa area needs further explanation, as Goetze assigned them to "northern" (YBC 6967) and "Uruk" (VAT 8520) provenances [Neugebauer & Sachs 1945, 149–150], followed by Høyrup [1999b, 26–30]. VAT 8520 belongs to an orthographically, terminologically, and museologically coherent group of 14 tablets (Goetze's "4th Group") which has spelling conventions similar to those of his "3rd Group," most of whose 14 members were said by dealers to come from Uruk [Neugebauer & Sachs 1945, 149 n. 356]. VAT 8512, the only provenanced tablet of "Group 4," however, was claimed by its dealer to originate in Larsa [Neugebauer 1935–1937, I 340]. Whichever attribution is correct—and neither is entirely watertight—Uruk (modern Warka, 31°18′ N 45°40′ E [Roaf 1990, 232]) is less than 25 km upstream from Larsa on the ancient Iturungal canal off the Euphrates river. Uruk was part of the Larsa polity between 1804 and 1762 BCE, when both cities were captured by Hammurabi's Babylon [Roaf 1990, 112; Kuhrt 1995, 109] and is certainly to be counted as within the environs of Larsa (Criterion 2).

Goetze's "5th Group," of which YBC 6967 is one of just three members [Neugebauer & Sachs 1945, 150], does not stand up to close scrutiny, however [cf. Høyrup 1999b, 26]. YBC 6967 has much in common with "4th Group" tablets, in both orthography (only the writing *ú-su-uh* for *usuh* "subtract" (rev. 2) differs from "4th Group" *ú-sú-uh* [cf. Neugebauer & Sachs 1945, 149–150]) and terminology (sharing the use of the verb *nasāhum* for subtraction, syllabically written *mi-nu-um* "what?" introductory *at-ta* "you," the word *takiltum* which is discussed further below, and the treatment of ÍB.SI$_8$ "square (root)" as a noun). Høyrup particularly associates many of these terminological features with his "possible subgroup 4B" comprising VAT 8512, YBC 8600, and YBC 8633 [Høyrup 1999b, 27–30]. It is, in short, more likely that YBC 6967 is to be provenanced in the Uruk–Larsa region than in some ill-defined "northern" area.

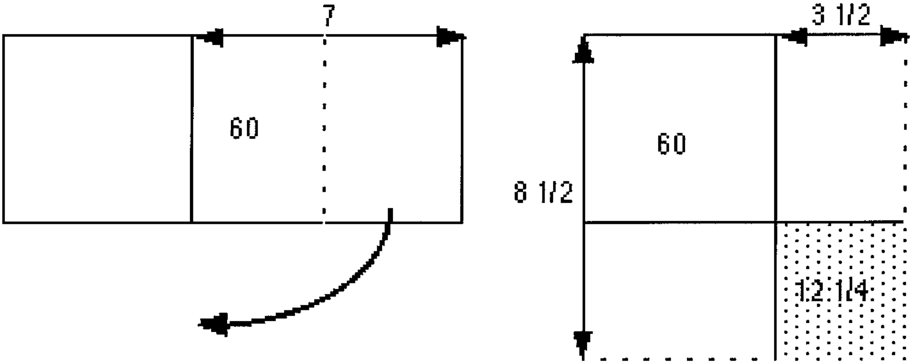

FIG. 4. "Completing the square" for YBC 6967 (drawing by the author).

> [A reciprocal] exceeds its reciprocal by 7. What are [the reciprocal] and its reciprocal? You: break in half the 7 by which the reciprocal exceeds its reciprocal, and 3;30 (will come up). Multiply 3;30 by 3;30 and 12;15 (will come up). Append [1 00, the area,] to the 12;15 which came up for you and 1 12;15 (will come up). What is [the square-side of 1] 12;15? 8;30. Put down [8;30 and] 8;30, its equivalent, and subtract 3;30, the *takiltum,* from one (of them); append (3;30) to one (of them). One is 12, the other is 5. The reciprocal is 12, its reciprocal 5.

The product of a reciprocal pair is, by definition, 1 (or any other power of 60). The problem-setter here has assumed that reciprocals are the same order of magnitude, and that their product is 60, in order to get an integer difference of 7 between the two unknowns. The problem is solved by "completing the square" (Fig. 4), with the instructions given in a very concrete fashion.

The two unknown reciprocals delimit a rectangle of area 1 00. The difference between them is measured off on the long side of the rectangle, and halved. This portion of the rectangle is torn off and pasted on underneath, to form an L-shaped figure of the same area as before. But the two inner arms of the L now define a square of length 3;30 (and area 12;15). The outer arms of the L must therefore describe a square of area 1 00 + 12;15 = 1;12;15. This large square must have length 8;30, so the lengths of the original rectangle (our unknown reciprocals) must be 8;30 + 3;30 = 12 and 8;30 − 3;30 = 5 respectively.

What does this have to do with Plimpton 322? The solution to YBC 6967 describes the formation of a large square from the juxtaposing of a small square space and a shape of area 1 00. If we redraw this area 1 00 as a middle-sized square with length $\sqrt{1\ 00}$ we have a particular case of $d^2 = b^2 + l^2$. In other words, in the course of solving a problem about finding a reciprocal pair given their integer difference, we have generated a "Pythagorean" triple: 8;30, $\sqrt{60}$, and 3;30. Admittedly this is a rather difficult triple to deal with in Old Babylonian arithmetic, because the second element is irrational ($2\sqrt{15}$) and susceptible only to approximate calculation. But if we revert to the normal definition of reciprocal pairs, whereby their product is 1 (or some even power of 60), then our method will serve to generate "Pythagorean" triples nicely.

If we make the necessary adjustments to the reciprocal pair in YBC 6967—namely defining their product as 1, not 60—we get a difference of 12 – 0;05 = 11;55. Half of this is 5;57 30, our smaller square-side. Square this number and add 1, to give 36;30 06 15. Find the square-side: 6;02 30.[22] So we have generated the triple 6;02 30, 5;57 30, and 1. We can now multiply out by products of 2, 3, and 5 to get a triple with the shortest possible strings of numerals:

6;02 30	5;57 30	1	× 2 (because the first two numbers terminate in 30) =
12;05	11;55	2	× 12 (because the first two numbers terminate in 5) =
2 25	2 23	24	

Or we can equally well take 5 − 0;12 = 4;48 as our starting point. Half of 4;48 is 2;24. Square this and add 1 to give 6;45 36. Its square-side is 2;36. So our triple is 2;36, 2;24, and 1. We can try multiplying out again:

2;36	2;24	1	× 5 (because the first two numbers terminate in products of 12) =
13	12	5	

Neugebauer and Sachs [1945, 41] had already pointed out that it was possible to generate the numbers found in Plimpton 322 from reciprocal pairs, and they listed the first four. (They did not believe, however, that it was the method used; the argument is taken up again in the following section.) The full set of calculations is given in Table 6. The first two columns contain the reciprocal pairs; the third and fourth their semi-differences and semi-sums (corresponding to the 3 1/2 and 8 1/2 in Fig. 4). The fifth column (Column I on Plimpton 322 as extant) contains the areas of the large squares, found (if, as in YBC 6967, the reciprocals are unknown) by adding 1 to the square of the semi-difference, or (if the reciprocals are known) by squaring the semi-sum. The next two columns show the two square-sides (small and large) multiplied out by successive factors to get the shortest possible strings (Columns II–III of Plimpton 322), while the penultimate one shows the products of those factors, namely the corresponding long sides. As in Column IV of the tablet, the final column simply serves as a line-counter.

EARLIER PROPONENTS OF THE RECIPROCAL THEORY

The theory set out here is not new and I certainly would not want to claim it for myself. It was first proposed by Bruins [1949; 1955] soon after the tablet was published, then reappeared in various guises some 25 years later in three apparently independent studies by Schmidt [1980]; Voils *apud* Buck [1980]; and Friberg [1981], although none had the supporting linguistic and conceptual evidence cited here but presented it in modernising algebraic form like Neugebauer's p, q theory. So why has it been largely ignored by the authors of generalist histories of mathematics, and why should it no longer be?

Bruins faced two major obstacles to the acceptance of his theories: he had extraordinarily difficult personal relationships with other scholars [Høyrup 1996, 15] and at times made startlingly sloppy mistakes [e.g., Fowler & Robson 1998, 375 n. 14]. His writing style is often almost incomprehensible and riddled with sweeping generalisations, exclamation marks, and venomous hyperbole. For instance, Bruins [1955, 118] claims, outrageously, that Plimpton 322 is a complete tablet in order to justify his contention that the first column does not begin with 1s. In the same article [1955, 119] he claims that *all* 4-place reciprocal

[22] See Fowler & Robson [1998] for a discussion of the standard OB method for finding square-sides. This large square-side is in fact simply the average or semi-sum of the two reciprocals involved, as can be seen in Fig. 4, so there would have been no need to actually perform the extraction if the reciprocals were known.

TABLE 6
From Reciprocal Pairs to Plimpton 322 Entries

x	$1/x$	$(x - 1/x)/2$	$(x + 1/x)/2$	$\{(x + 1/x)/2\}^2$	Short side	Diagonal	Long side	Line
2;24	0;25	0;59 30	1;24 30	1;59 00 15	1 59	2 49	2 00	1
2;22 13 20	0;25 18 45	0;58 27 17 30	1;23 46 02 30	1;56 56 58 14 50 06 15	56 07	1 20 25	57 36	2
2;20 37 30	0;25 36	0;57 30 45	1;23 06 45	1;55 07 41 15 33 45	1 16 41	1 50 49	1 20 00	3
2;18 53 20	0;25 55 12	0;56 29 04	1;22 24 16	1;53 10 29 32 52 16	3 31 49	5 09 01	3 45 00	4
2;15	0;26 40	0;54 10	1;20 50	1;48 54 01 40	1 05	1 37	1 12	5
2;13 20	0;27	0;53 10	1;20 10	1;47 06 41 40	5 19	8 01	6 00	6
2;09 36	0;27 46 40	0;50 54 40	1;18 41 20	1;43 11 56 28 26 40	38 11	59 01	45 00	7
2;08	0;28 07 30	0;49 56 15	1;18 03 45	1;41 33 45 14 03 45	13 19	20 49	16 00	8
2;05	0;28 48	0;48 06	1;16 54	1;38 33 36 36	8 01	12 49	10 00	9
2;01 30	0;29 37 46 40	0;45 56 06 40	1;15 33 53 20	1;35 10 02 28 27 24 26 40	1 22 41	2 16 01	1 48 00	10
2	0;30	0;45	1;15	1;33 45	45	1 15	1 00	11
1;55 12	0;31 15	0;41 58 30	1;13 13 30	1;29 21 54 02 15	27 59	48 49	40 00	12
1;52 30	0;32	0;40 15	1;12 15	1;27 00 03 45	2 41	4 49	4 00	13
1;51 06 40	0;32 24	0;39 21 20	1;11 45 20	1;25 48 51 35 06 40	29 31	53 49	45 00	14
1;48	0;33 20	0;33 20	1;10 40	1;23 13 46 40	28	53	45	15

pairs between 2 24 and 1 48 are accounted for in the tablet; they are not (see below). On the same page he claims that the square roots in Columns II and III would have been found by inspection for common square factors. Not only is this an extremely impractical method,[23] but for historical support it depends on a tablet that he had published extremely badly and incompletely the previous year [Bruins 1954, 56].[24] Again Bruins [1955, 119] tries to wring the meaning "reduced value" out of the word *mithartum* (cf. above)—and all this in a poor print-quality Iraqi journal available only in a few specialist Assyriological libraries. None of these factors were likely to encourage his colleagues to treat his proposal seriously.

Schmidt [1980] got close to the scenario supported here—arguing in effect that Plimpton 322 concerns not Pythagorean triples *per se* but the sums of reciprocal pairs scaled up by a factor *l*—but presented perhaps the weakest historical analysis. For a start, it is apparent that the subject of his study is not an Old Babylonian tablet but an error-free table of Indo-Arabic numbers in the sexagesimal system. Plimpton 322 is first described solely as "a text containing Pythagorean numbers" [Schmidt 1980, 4] with no indication of its date, provenance, or even physical appearance and is referred to thereafter as "the Plimpton text" [Schmidt 1980, 4 and *passim*]. Indeed, "the Plimpton text in its present form" [Schmidt 1980, 4] is shown in his Table 1 on the following page. Its headings consist solely of the Roman numerals I to IV, under which are four rows of data, three rows of ellipses, and a final row of data. His message is that the material, historical form of Plimpton 322 is irrelevant.

Later in the article, when Schmidt wishes to provide further support for his adaptation of Bruins' reciprocal theory, he states that "[s]uch a system actually occurs in the text material (Neugebauer, *Mathematische Keilskrift-Texte* [sic], vol. I, p. 106)" [Schmidt 1980, 9]. It is not the museum number and publication details of a cuneiform tablet that he cites, but a German "text" of the 1930s. Neugebauer [1935–1937, I 106], to which he refers us, is neither transliteration nor translation of a Mesopotamian mathematical tablet, nor even a cuneiform copy or photograph, but a mathematical discussion headed "Quadratische Gleichungen für reziproke Zahlen (Rs. 10 bis 27)." Turning back ten pages to the start of the chapter, we discover that the discussion concerns four problems on AO 6484, a tablet from Uruk purchased by the Louvre—and that its date is "Seleukidisch" [Neugebauer 1935–37, I 96].[25] In other words, Schmidt admits as support for his argument mathematics from the Seleucid (i.e., Hellenistic) period, roughly a millennium and a half later than the date of Plimpton 322 and younger even than the *Elements*.[26]

[23] I tried it for all values on Plimpton 322: it is virtually impossible. To factor out squares by inspection, as Bruins wants, one has to choose factors at each stage, test them, and backtrack if they are wrong (which in my case they often were). On the other hand, to find reciprocals using "The Technique," (page 29) or to find square-sides (above), one chooses a suitable starting approximation and continues until done: it does not really matter how accurate the approximation is, or whether it is an over- or underestimate; one gets there in the end. After the initial choice no further decision needs to be taken, except when to stop.

[24] The tablet, IM 54472 (unprovenanced), is published without photograph or copy but solely in defective transliteration so that it is impossible to determine which cuneiform signs were used to write the text; the shape, size, and physical condition of the tablet; or how accurate or plausible Bruins' reading of it is.

[25] In fact the scribe named in its colophon, one Anu-aba-utē̄r, is known to have been active in the early second century BCE [cf. Høyrup 1990a, 347 n. 180].

[26] Nevertheless, as Jens Høyrup (personal communication) comments, AO 6484 provides interesting evidence that "the igi–igibi problem-type, as the only certain survivor from OB school 'algebra' must have been more important [or pervasive?] than ascertained by the two surviving texts taken alone". Cf. Høyrup [1990a, 347–348 n. 183] on the arithmetical terminology and conceptualisation of the reciprocal-pair problems in AO 6484.

Schmidt's ignorance of the historicity of his subject matter pervades the article. He does not engage with, or even acknowledge the existence of, the errors on the tablet, while the headings (or rather the two words "width" and "diagonal") are mentioned only in the very last line [Schmidt 1980, 13] during an unconvincing last-minute attempt to fit them into his interpretation. His final reconstruction is a table eight or nine columns wide [Schmidt 1980, 13]—which would mean restoring to Plimpton 322 a total width of 25–30 cm and a sharply asymmetric curvature. (Schmidt, though, does not register the physical implications of his theory.)

He is, however, able to give some internalist, mathematically orientated justifications for his proposals:

> We notice that in the explanations given by Neugebauer and Sachs, and E. M. Bruins the number l [$\cdots$] plays a rather significant role. In our explanation the number l does not occur explicitly. Nor does l occur explicitly in the text proper, and in this respect our explanation agrees better with the text than the two previous explanations. [Schmidt 1980, 10–11]

Voils' work, although signaled by Buck [1980, 344] to appear in *Historia Mathematica*, never actually made it into print, so it is not always possible to disentangle his theory from Buck's (mis)interpretation of it. Like Schmidt, Buck and Voils propose that Plimpton 322 concerns the sums of regular reciprocal pairs x and x^{R}:

> [T]he entries in [in Columns I–III of] the Plimpton tablet could have been easily calculated from a special reciprocal table that listed the paired values x and x^{R}. Indeed the numbers [in Columns II–III] can be obtained from $x \pm x^{R}$ merely by multiplying these by integers chosen to simplify the result and shorten the digit representation. [Buck 1980, 344]

But, again like Schmidt, Buck and Voils have difficulty in handling the historical data. The unidentified mathematical problem they present (which is presumably YBC 6967) is mistakenly provenanced to the city of Nippur. More seriously, they fail to state that no such "special reciprocal table" actually exists or to show how such a reciprocal table might have been calculated. Moreover, while pointing out that the p, q theory fails to explain the presence, position, and contents of Column I on the tablet [Buck 1980, 343], they cannot give a convincing justification for the presence of Columns II to III and once more do not attempt to deal with the headings.

The following year Friberg [1981a] produced the most detailed analysis of the tablet to date. His article stands on the cusp of the old era and the new: on the one hand fiercely mathematical and modernising with regard to the content of the tablet, and on the other extremely perceptive about authorial intention. He summarises his theory as follows, with a, b, and c equivalent to our l, b, and d:

> It is easy to verify that the listed values [in Plimpton 322] . . . are precisely the ones that can be obtained by use of the triangle parameter equations
>
> $$b = a\bar{b}, \quad c = a\bar{c}; \quad \bar{b} = \frac{1}{2}(t' - t) \quad \bar{c} = \frac{1}{2}(t' + t)$$
>
> if one allows the parameter t (with the reciprocal number $t' = 1/t$) to vary over a conveniently chosen set of 15 rational numbers $t = s/r$, and if the multiplier a is chosen in such a way that b and c become integers with no common prime factors. [Friberg 1981a, 277]

Like the theories of Schmidt and Buck/Voils, this explanation boils down to using reciprocal pairs to generate the table. But Friberg recognises that the reciprocals must themselves have been generated and sets out to do so himself. Rather than searching the OB mathematical corpus for a toolkit of culturally appropriate techniques (cf. pages 26–29) he simply shows us how he has done it:

> By writing the parameter t and its reciprocal t' as $t = s/r$ and $t' = r/s$, and by letting the pair (r, s) vary over all admissible parameter pairs (coprime pairs of regular sexagesimal integers) within a bounded "strip" in the (r, s) plane, one can generate an arbitrarily large set of parameter values in a systematic and straightforward way. This type of procedure has been followed in the construction of the table in Figure 2.2. [Friberg 1981a, 288]

"Figure 2.2" [Friberg 1981a, 286] is a log–log graph! This is "rationalist reconstruction" at work: nowhere in this discussion does Friberg acknowledge that an OB scribe may have had very different ideas about what constituted a "conveniently chosen" set of numbers. In contrast, he then goes on to announce that the contents of Columns I–III

> can be obtained fairly easily from the values of b and c, using only methods that would have been available also to a mathematician of the Old Babylonian period [Friberg 1981a, 289],

so it is not clear why he did not take this approach to the reciprocal pairs too.

After a heavily algebraised but mostly successful account of possible arithmetical techniques behind, and therefore the nature of the errors in, Plimpton 322, Friberg moves on to tackle "the purpose of the text" [Friberg 1981a, 299–306]. Like all his immediate predecessors he is not equipped to deal with the internal evidence provided by the headings [Friberg 1981a, 300] and like Schmidt inappropriately cites the four reciprocal-pair problems of the Hellenistic tablet AO 6484 as his primary external evidence [Friberg 1981a, 303]. The paper concludes with lengthy "reflections on the origins of the 'Pythagorean theorem'" [Friberg 1981a, 306–315], working through all Mesopotamian evidence then known for problems about right-triangles.

In sum, the reciprocal theory as presented by Schmidt, Buck/Voils, and Friberg had no apparent advantages over the mainstream p, q interpretation. Although it purported to be more consistent with OB mathematical culture (Criterion 2), the historical evidence was handled clumsily. Schmidt in particular was chronologically and artefactually insensitive (cf. Criteria 1 and 4). The non-numerical contents of the tablet were practically ignored (cf. Criterion 5), and while the existence and placement of the errors (Criterion 3) and of Column I could be justified, there was now no satisfactory explanation for Columns II–III (cf. Criterion 6). On the face of it, there was little to choose between reciprocals and p, q generators—except that the latter theory had the authority of Neugebauer behind it.

So what does make Bruins' reciprocal theory more convincing than the standard p, q generating function—or, indeed, the trigonometric table? I have already showed that its starting points (reciprocal pairs, cut-and-paste algebra) and arithmetical tools (adding, subtracting, halving, finding square sides) are all central concerns of Old Babylonian mathematics: it is sensitive to the ancient thought-processes and conventions in a way that no other has even tried to be. For example, in this theory the values in Column I are a necessary step towards calculating those in Column III and may also be used for Column II. And the Column I values themselves are derived from an ordered list of numbers. In other words, as it has been presented so far, the theory satisfies Criteria 1, 2, 3, and 6 reasonably well. But there remain several outstanding concerns to be answered:

- What of the damaged heading in the first extant column?
- Do the controversial 1s in Column I exist or not?
- Can the errors in the tablet be explained?
- How would the reciprocal pairs have been found or known?
- Are any entries missing from the tablet, and why?
- What would have been in the missing portion of the tablet?
- What are Columns II and III for?
- Why was Plimpton 322 written?

Let us deal with those questions in the order given.

COLUMN I: THE WORDS AND THE ONES

So, what of the mystery heading over the first column and the controversial 1s? We know (page 8) that the 15 rows beneath it contain *either* the ratio of the square on the short side to the square on the long side *or* the ratio of the square on the diagonal to the square on the long side (depending on whether you believe in the existence of the 1s at the beginning of the column or not). According to our reciprocal theory (cf. Fig. 4), this is equivalent to the area of the imaginary small square or the area of the large square (composed of 1 plus the small square).

Neugebauer and Sachs read the heading as

> [*ta*]-*ki-il-ti ṣi-li-ip-tim* / [*ša in*]-*na-as-sà-hu-ú-ma* SAG *i-...-ú*
> The *takiltum* of the diagonal which has been subtracted such that the width . . . . [Neugebauer & Sachs 1945, 40]

Now, *ṣiliptum* "diagonal" (here in the genitive case) and the Sumerogram SAG for *pūtum* "short side" we have already met in the Column II and III headings. Between them are the relative pronoun *ša* "who, which, whose, whom," and the verb *innassah* "is torn out" from the passive stem of the verb *nasāhum* 'to tear out'. The verb has two suffixes: the subjunctive *-u* (governed by the relative *ša*) and the conjunction *-ma* "and then," "so that." We are left with two difficult words: a noun in the construct at the beginning of the heading and a subjunctive verb in the third person at the end. Because *pūtum* is written as a logogram with no case ending, we cannot tell if it is the subject or object of the mystery verb. The eagle-eyed reader will have already spotted 3;30 *ta-ki-il-tam* in YBC 6967 (p. 17), where, in the accusative case, it refers to the length of the small square which is imagined in the lower right corner of the rearranged rectangle in Fig. 4. But according to our reciprocal theory the contents of Column I are all squares, not lengths. This is not in itself a problem, as we have already seen that geometrical configurations share their names with their defining components, and in particular that squares and their square-sides are named identically. So it is to be expected that the *takilti* in Plimpton 322 might refer to the area of a square instead of a square-side. But we still have a difficulty: in YBC 6967 the *t*.-square is the little one, of the short side, whereas in Plimpton 322 the *t*.-square of the diagonal can only be the large, composite square. And that means that we have to restore the 1s at the start of the line, as the remains of the tablet suggest we should.

But even if we accept that *takiltum* can refer to both large and small square we need to make an adjustment to our heading, because clearly the big square cannot be "torn out" of anything. If, on the other hand, we insert a vertical wedge for "1" into the damaged part of the tablet, and restore[27]

> [*ta*]-*ki-il-ti ṣi-li-ip-tim* / [*ša* 1 *in*]-*na-as-sà-hu-ú-ma* SAG *i-...-ú*
> The *takiltum* of the diagonal from which 1 is torn out, so that the short side . . .

we have a meaningful (and grammatically correct) first clause: subtract 1 from the *t*.-square of the diagonal and you will get the *t*.-square of the short side, from which the short side itself can be found.

What does *takiltum* actually mean? Huber [1957, 26] tentatively translated it "Hilfzahl" (helping number), implicitly deriving it from the verb *takālum* "to help" as the feminine participle *tākiltum* "helper." This is, on the face of it, a very attractive proposition. But the authoritative *Akkadisches Handwörterbuch* [von Soden 1959–1981, 1306] reads *takiltum* (no macron) and translates "bereitstehende (Verfügungs-) Zahl" ("available number"), while the extremely thoughtful Høyrup [1990a, 49, 264] reads *takīltum* (with a macron on the *i*) without translating. Both identify it as a nominal derivative of the verb *kullum* "to keep, hold." As might be expected, von Soden's reasons are primarily philological (and highly technical), whereas Høyrup focuses on (similarly technical) mathematical arguments. But they can both be reduced to the fact that the OB verb "to multiply geometrically" (i.e., to construct a rectangle or square from two perpendicular lines, as in Fig. 4) is probably the causative reciprocal form of *kullum*, and *takiltum* (a constructed square) can only be derived from the same root. The form of the noun is causative too; it means "something that has been caused to hold (something)"—which, as we would expect, suggests that the *takiltum* is conceptualised as a square configuration or frame rather than an area or a length. It is difficult to find a meaningful one-word English translation which takes into account both the mathematical context and the semantic constraints of its grammatical form, but for the moment I suggest "holding (-square)," pending a better suggestion.[28] All that remains is the last verb, whose meaning we can guess at, even if it is difficult to read—the scribe has tried unsuccessfully to squash into the end of the first column and it has ended up spilling into the second line of Column II. The word must signify something like "results," the standard OB mathematical terms for which are *tammar* "you see," the second person present tense of *amārum*" "to see"; and *illi*, the third person present of *elûm* "to be(come) high" which is normally translated "comes up" in this context. We can immediately exclude the first of these options, as all second person verbs are prefixed *ta*- (which looks nothing like the *i*-shown clearly on the tablet). We can explain the certain final -*ú* as another subjunctive marker governed by the relativiser *ša* that we have already restored, so we are now looking for traces on the tablet that would fit with *illiu* or contracted *illû*.

Legitimate syllabic spellings are shown in Fig. 5.[29] Both *il* and *li* are long signs, too long to together fit the traces on the tablet (a); while *lu* alone is too simple a sign to account for all

[27] This restoration was suggested first by Bruins [1949; 1967, 38], even though he did not believe in the existence of the initial 1s! It was also discussed by Price [1964, 8] who, hampered by his ignorance of Akkadian, wanted to understand SAG as "long side" and *nasāhum* as "to select," both of which meanings are impossible.

[28] The word *takiltum* is also found with the same meaning in two tablets from the Larsa region: VAT 8512: I 19 [Neugebauer 1935–1937, I 341], and VAT 8520 (ii): II 21 [Neugebauer 1935–1937, I 346] (cf. note 21).

[29] We can exclude spellings with the southern il_5 (=EL) because of the spelling [*ta*]-*ki-il-ti* in the line above. We can also exclude *il-li-ú* and *il-lu-ú* because of the unquestionably present *i*- at the beginning of the word. Spellings such as *i-il-ú* break the conventions of Akkadian spelling and are not an OB phenomenon.

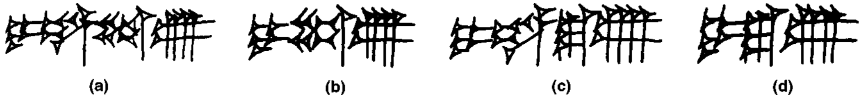

FIG. 5. Legitimate syllabic spellings of *illû*, subjunctive "comes up": (a) *i-il-li-ú*, (b) *i-li-ú*, (c) *i-il-lu-ú*, (d) *i-lu-ú* (drawings by the author).

the visible wedges (d), and *li* as written in *ṣiliptim* does not fit them either (b). This leaves us with spelling (c), and close inspection of the original tablet (or the photo in Neugebauer & Sachs [1945, pl. 25]) shows this to be a good fit, with the *il* and the *lu* squeezed into the end of the line, as shown in Fig. 6. In fact, this reading was first proposed by Goetze *apud* Price [1964, 8] but not fully explained there. So now we have a grammatically and mathematically meaningful heading for Column I,

> [*ta*]-*ki-il-ti ṣi-li-ip-tim* / [*šǎ* 1 *in*]-*na-as-sà-hu-ú-ma* SAG *i-il-lu-ú*.
> The holding-square of the diagonal from which 1 is torn out, so that the short side comes up,

and have thereby fulfilled Criterion 5.

ACCOUNTING FOR THE ERRORS

Now we come to the three nontrivial errors. The square in row II 13 can easily be accounted for if we assume that Column II was derived from Column I, as the latter's own heading explicitly states. The scribe forgot, in this one instance, to find the square side after subtracting 1 from the entry in Column I. So he multiplied the *square* of the short side and the *length* of the diagonal by successive regular factors to get the numbers shown on the tablet. This conclusion has implications for our interpretation; we shall return to it later.

Both of the remaining errors (double or half the correct value in line 15, and 3 12 01 for 1 20 25 in III 2) can be explained—as Bruins [1955] and Friberg [1981a] both concluded—by independent factorisation gone too far. That is, having found the basic triples (0;37 20, 1, and 1;10, 40 for line 15, and 0;58 27 17 30, 1, and 1;23 46 02 30 for line 2), the scribe multiplied up each number in the triple to eliminate common factors. For line 15 the calculation should have been:

0;37 20	1	1;10 40	× 3 because two numbers terminate in multiples of 20
1;52	3	3;32	× 5 because two numbers terminate in 2
9;20	15	17;40	× 3 because two numbers terminate in 20
28	45	53	

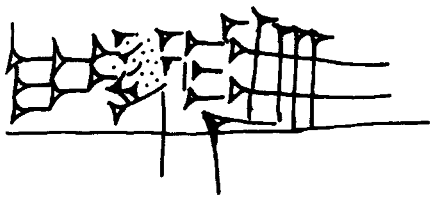

FIG. 6. The last word of the Column I heading, twice original size (drawing by the author).

but the scribe apparently doubled the resulting triple in an attempt to further reduce the common factors (to 56, 1 30, 1 46), and forgot to return to the first of these values when he saw the result was not satisfactory. Similarly, in line 2,

0;58 27 17 30	1	1;23 46 02 30	× 2 because two numbers terminate in 30
1;56 54 35	2	2;47 32 05	× 12 because two numbers terminate in 5
23;22 55	24	33;30 25	× 12 because two numbers terminate in 5
4 40;35	4 48	6 42;05	× 12 because two numbers terminate in 5
56 07	57 36	1 20 25.	

Again the scribe, not certain that he has found the optimum triple, continues to multiply by 12:

56 07	57 36	1 20 25	× 12
11 13 24	11 31 12	16 05 00	× 12
2 14 36 48	2 18 14 24	3 13 00 00	

and on reverting to the shortest triple forgets to convert the last of the three. Because the calculation has been done very roughly [cf. Robson 1999, 66, 245–277], he misreads the final three wedges of 3 13 and transfers 3 12 01 onto the good copy.

We have now reasonably satisified Criterion 3.

RECIPROCAL PAIRS AND MISSING ROWS

To judge from the curvature of the extant part of Plimpton 322 [cf. Friberg 1981a, 283 Fig. 1.3], there is probably room for two columns of roughly the width of Columns II and III in the missing portion, thereby adding no more than about 5 cm to the width of the tablet. Let us call them Columns A and B. Let us suppose for the moment that they contained the reciprocal pairs listed in the first two columns of Table 6. As already mentioned, Neugebauer and Sachs were not in favour of the reciprocal pairs playing any role in generating the table. They stated:

> [O]ne can also produce Pythagorean numbers by using one parameter α and its reciprocal $\bar{\alpha}$ where $\alpha = p/q$. But a comparison of the [first] four lines [of pairs] shows immediately that neither α nor $\bar{\alpha}$ could have been the point of departure but only the simple numbers p and q. [Neugebauer & Sachs 1945, 41]

Their argument is, I think, based on the fact that the reciprocal pairs are up to four sexagesimal places long while they knew almost exclusively of tables listing one- and two-place pairs for the Old Babylonian period [e.g., Neugebauer & Sachs 1945, 11–12]. Just four known exceptions[30] generated reciprocal pairs up to 8 or 9 sexagesimal places long by successive halving and doubling of two-place pairs easily derivable from the standard tables.

This method of finding nonstandard reciprocal pairs is generalised somewhat in Str 366 (ii) from Uruk [Neugebauer 1935–1937, I 257–259; Sachs 1947, 235]: it consists of dividing the (regular) number given by a (regular) factor—in this case 3—so that a number in the

[30] UM 29-13-21 [Neugebauer & Sachs 1945, 13–15], CBS 10201, BM 80150, and VAT 6505 [Neugebauer 1935–1937, I 23–24; I 49–50; III 52]—the first two from Nippur, the third from Sippar, the latter unprovenanced.

standard reciprocal table is reached. Its reciprocal is then multiplied by that same factor to produce the reciprocal of the original number.

Very shortly afterwards, Sachs published a sizable body of Old Babylonian tablets demonstrating another method (which he dubbed "The Technique") for finding the reciprocals of regular numbers [Sachs 1947].[31] It can be summarised algebraically (and anachronistically) in the following way. A sexagesimally regular number c has to terminate with some one- or two-place integer, a, in the standard list (see Table 5). Call the difference between them b, so that $c = b + a$. Therefore,

$$\begin{aligned}\frac{1}{c} &= \frac{1}{b+a}\\ &= \frac{1}{a}\cdot\frac{1}{1+b/a}.\end{aligned}$$

Explicit instructions for using "The Technique" are given on the unprovenanced OB tablet VAT 6505 [Neugebauer 1935–1937, I 270–273; Sachs 1947, 226–227], and many OB examples of its use are now known [Robson 2000a, Table 3]. Let us take 2;22 13 20, from the second pair in Table 6, as an example of how it works. First, pick out an entry a from the standard reciprocal table from the end of the number. We could choose 20, but let us be more ambitious and set $a = 0;02\ 13\ 20$, leaving $b = 2;20$. The reciprocal of 0;02 13 20 is 27. Multiply this by b to give 1 03. Add 1—1 04—and take the reciprocal: 0;00 56 15. (In our case the pair is in the standard table, but in other cases (and there are OB examples known) one needs to iterate, taking this last value as a new c.) Multiply 0;00 56 15 by 27 to get $0;25\ 18\ 45 = 1/c$, as wanted.

So, although there are no four-place tables of reciprocals known, it was easily within the abilities of an Old Babylonian scribe to generate regular reciprocal pairs and to sort them in order.[32] And that ascending numerical order in the first column, the universal method for sorting OB tables, accounts for the fact that the extant Column I is ordered while Columns II and III are not.

We are now moving toward a fuller response to Criterion 6. But we have not actually addressed the question of how the scribe would have found suitable four-place regular integers, whose reciprocals he could then calculate. This matter is related to the question of whether or not the table is full. Trivially, it is not: the reverse of the tablet has been ruled vertically in preparation for a continuation of the table. Less trivially, it is a more difficult matter to determine whether or not there are omissions from the sequence as we have it.

Most earlier analyses of the tablet [e.g., Friberg 1981a, 284–289] have started from the assumption that the tablet is exhaustive and have attempted to determine the criteria by which the scribe chose his starting points, whether p, q generators or reciprocal pairs. All of these attempts seek a single rule for choosing those starting points (e.g., Friberg's

[31] It had already been treated briefly by Neugebauer [1935–1937, III 52].

[32] One example of this technique (*UET* 6/2 295, from Ur [Robson 1999, 251]) was certainly carried out by a trainee scribe: the tablet is a characteristic round "rough jotter," on the obverse of which is a Sumerian proverb—a typical OB school exercise. The tablet was found with many hundreds of other school documents.

"restrictions on the parameters" [1981a, 284]). But are we justified in assuming that the concept of mathematical completeness would have meant anything at all in the early second millennium BCE? Or that the scribe must have generated his starting numbers (p, q, or reciprocals) using a single algorithm? And does it really matter? That depends on one's point of view, specifically on whether ancient authorial intention or modern audience reception has priority. If we are concerned with Plimpton 322 as an aesthetically pleasing piece of (implictly modern) mathematics then it apparently does matter that the table is complete and elegantly generated: indeed, these assumptions have never been questioned. But we are thereby indulging in mathematical criticism, not history. More seriously, we are guilty of acting like Pingree's "treasure hunters seeking pearls in the dung heap," privileging the apparently modern at the expense of the obviously ancient.

If on the other hand we are interested in what Plimpton 322 might have been for, then its degree of completeness is an issue. Did it matter to its ancient compiler? Would it even have been a meaningful issue for him? I would argue, almost certainly not. The 40 standard OB multiplication tables, for instance, provide us with an informative parallel. They are complete neither in themselves (with multiplicands 1–20, 30, 40, and 50) nor as a set (the complete set of multipliers is shown in Table 7). Nor does that set appear to have been chosen according to any one criterion or generating algorithm: many are one-place sexagesimally regular numbers (but 27, for instance, is missing); many are significant numbers in the

TABLE 7
The Standard Set of OB Multipliers[a]

Multiplier	Decimal equivalent					Multiplier	Decimal equivalent				
50	50	I	D	R		8	8	I		R	C
48	48	I		R	C	7 30	450			R	C
45	45	I		R	C	7 12	432				C
44 26 40	160,000			R		7	7				
40	40	I		R	C	6 40	400		D	R	C
36	36	I			C	6	6	I		R	C
30	30	I		R	C	5	5	I		R	C
25	25	I	D	R		4 30	270				C
24	24	I		R	C	4	4	I		R	C
22 30	1,350					3 45	225			R	C
20	20	I		R	C	3 20	200		D	R	C
18	18	I		R	C	3	3	I		R	C
16 40	1,000		D			2 30	150		D	R	C
16	16	I		R	C	2 24	144			R	C
15	15	I		R	C	2 15	135				C
12 30	750		D		C	2	2	I		R	C
12	12	I		R	C	1 40	100		D	R	C
10	10	I		R	C	1 30	90			R	C
9	9	I		R	C	1 20	80			R	C
8 20	500		D		C	1 15	75		D	R	C

[a] I = 1-place regular integer; D = round decimal number; R = entry in the standard reciprocal table (cf. Table 5); C = commonly occurring coefficient (technical constant) (cf. Robson [1999, 325–332]).

decimal system (but 4 10 [=250], for instance, is missing).[33] Many are also found in the standard reciprocal table (Table 5) (but 2 13 20, for instance, is missing); and many too are commonly occurring coefficients listed in the tables of technical constants [cf. Robson 1999, 325–332] (but 26 40, for instance, is missing). On the other hand, 22 30 belongs to none of these categories, and seems to have been included because it is half of 45. Similarly, 7 is there to complete the set of integers to 10. In sum, the set of multipliers appears to have been assembled because it gives good coverage of the numbers most likely to be used by scribes in their everyday arithmetical work.

Returning to Plimpton 322, we note that all the reciprocals are four-place or less, with the total number of places in each pair always less than seven. Under these selection conditions, it happens that three reciprocal pairs are missing from the table, as shown in Table 8, lines 4a, 8a, 11a. The three corresponding entries in the columns of Plimpton 322 are also given there, as well as the long side of the triangle and the analogous p, q generators.

Let us for a minute suppose that the scribe did not have access to a "generating function" [cf. Friberg 1981a, 284–289], but collected or omitted his reciprocal pairs using a variety of criteria just as the compiler(s) of the standard set of multiplication tables presumably had. Why would he have chosen the ones from which the table is derived, and omitted the three in Table 8? The reciprocal pairs are listed as integers in Table 9, with their decimal equivalents given for ease of reference. Let us assume that the scribe started off by choosing the two-place reciprocal pairs at the top and bottom of the table, which he knew (or had calculated) would yield nice Pythagorean triples. He now had to find as many pairs as he could between them, and chose, for ease of calculation, to restrict himself to four-place or shorter numbers (which would also result in four-place or shorter numbers in the final columns of the table). Naturally, only a few pairs in his range occur in the standard reciprocal list, but many more are derivable from it through the simple (and well-attested) expedient of successively halving and doubling standard reciprocal pairs, or tripling and dividing by 3. In this way, 11 pairs in our scribe's range are simply derived from the standard reciprocal table; others he may have picked from the standard multiplication tables, coefficient lists, or simply his working knowledge of the sexagesimal system. Only 4a and 8a, two of the three pairs he omitted, do not seem to have been easily derivable, using attested methods, from the basic OB arithmetical environment. The selecting and sorting processes need not have been concurrent; as we have seen, it was normal scribal practice to sort numerical data by size.

And we can also argue backwards, from Columns II and III. Looking at the short sides and diagonals and the implicit long sides (Table 1), we notice that no long side is more than two sexagesimal places long, and no short side or diagonal is more than two and a half sexagesimal places long—that is, with no tens in the largest of the three places. If we suppose that place-length was a desired attribute of these parameters and not simply coincidental, then neither 4a nor 11a should have been included: the short side and diagonal of 4a are half a place too long, while the long side of 11a has a whole place too many. Only

[33] There was a significant decimal substatum to the sexagesimal system: even in the discrete metrology of late fourth millennium Uruk 60s had been conceptualised as six bundles of 10 units [Nissen *et al.* 1993, 28], while "hundred" and "thousand" units were frequently used in administrative records from almost all periods and places [Friberg 1987–1990, 537]. There is particularly clear evidence from OB Mari that some scribes actually calculated decimally [Soubeyran 1984, 34].

TABLE 8
Possible Missing Lines in Plimpton 322

Line	x	$1/x$	$1 + \{(x - 1/x)/2\}^2$	Short side	Diagonal	Long side	p	q
4a	2;18 14 24	0;26 02 30	1;52 27 06 59 24 09	18 41 59	27 22 49	20 00 00	4;48	2;05
8a	2;06 33 45	0;28 06 40	1;40 06 47 17 32 36 15	3 55 29	6 12 01	4 48 00	2;15	1;04
11a	1;57 11 15	0;30 43 12	1;31 09 09 25 42 02 15	3 12 09	5 28 41	4 26 40	2;05	1;04
6a	2;10 12 30	0;27 38 52 48	1;43 49 39 19 46 33 57 36	1 25 28 01	2 11 32 49	1 40 00 00	10;25	4;48
9a	2;02 52 48	0;29 17 48 45	1;36 29 27 01 09 22 38 26	13 51 51	22 32 41	17 46 40	4;16	2;05
12a	1;53 46 40	0;31 38 26 15	1;28 06 37 40 40 47 15 56	13 08 31	23 16 01	19 12 00	4;16	2;15

TABLE 9
Criteria for Choosing the Reciprocal Pairs

Line	x	$1/x$	SRP[a]	Halved/doubled SRP	Tripled SRP	Entry in 7;12 mult. table	Coefficient	Decimal equivalents of x and $1/x$	
1	2 24	25	yes			=20 × 7;12	yes, —	144	25
2	2 22 13 20	25 18 45			×27 = 1 04, 56 15			512,000	91,125
3	2 20 37 30	25 36		×32 = 1 15, 48				506,250	1,536
4	2 18 53 20	25 55 12						**500,000**	93,312
4a	2 18 14 24	26 02 30						497,664	93,750
5	2 15	26 40		×4 = 9, 6 40	×3 = 45, 1 20		yes, yes	135	1600
6	2 13 20	27	yes				yes, —	8,000	27
7	2 09 36	27 46 40				=18 × 7;12		7,776	**100,000**
8	2 08	28 07 30		×2 = 1 04, 56 15			yes, —	128	101,250
8a	2 06 33 45	28 26 40						455,625	102,400
9	2 05	28 48	yes					125	1,728
10	2 01 30	29 37 46 40	no[b]					7,290	6,400,000
11	2	30	yes				yes, yes	2	30
11a	1 57 11 15	30 43 12		×64 = 2 05, 28 48				241,875	110,592
12	1 55 12	31 15				=16 × 7;12		6,912	1,875
13	1 52 30	32	yes				yes, —	6,750	32
14	1 51 06 40	32 24			×9 = 16 40, 3 36			**400,000**	1,944
15	1 48	33 20		×2 = 54. 1 06 40		=15 × 7;12	yes, yes	108	2000

[a] SRP = standard reciprocal pair; see Table 5.
[b] Attested on a tablet from OB Ur [Robson 1999, 272].

line 8a ought to have been in the table under this hypothesis—and this was one of the two reciprocal pairs which we have just seen was not easily found using OB methods.[34]

Is this explanation for the scribe's choices any less historically plausible than previous scholars' [e.g., Friberg 1981a 285–288], despite its shockingly *ad hoc* nature? We have to remember (cf. Criteria 1–2) that we are trying to (re)construct what a real human being, nearly 4000 years ago, might have thought and done to produce the figures on Plimpton 322, not an idealised mathematical automaton operating according to implicitly modern rules. I leave it for the reader to decide, with another reminder that historical plausibility is a completely different issue from mathematical aesthetics.

WHO WROTE PLIMPTON 322, AND WHY?

Having dismissed modern audience reception as the paramount factor in analysing ancient mathematics and attempted to show—in the light of current knowledge—how Plimpton 322 might be constructed, I am duty bound to address the question of ancient authorial intention: what was the tablet for?

For a start, there is no evidence that Plimpton 322 was intended to be any sort of definitive reference table. First, there are no ancient duplicates of the tablet, compared to the many hundreds of extant standard arithmetical and metrological tables and lists and the several dozen duplicates of other types of table, such as cubes or powers of 10 [cf. Neugebauer 1935–1937, I 4–96; II 36–37; III 49–51; Neugebauer & Sachs 1945, 4–36]. Second, (what we presume are) intermediate results are shown—in Column I—while only two of the three expected end-products are there: the length l is missing, against modern expectations. So whatever the scribe's aim, it was not simply to compile a complete list of Pythagorean triples.

Nor is it convincing to label Plimpton 322 as "research mathematics" a sophisticated exercise in manipulating numbers for no other purpose than to satisfy idle curiosity: for early second millennium Mesopotamia we have no evidence whatsoever for a leisured middle class of the kind whose members occasionally pursued mathematical recreations in Classical antiquity[35] or in early modern Europe. Four thousand years ago scribes had to work for a living, for the most part as bureaucrats and administrators for the big institutions—temples and palaces—but also entrepreneurially serving the documentary needs of private households and individuals. Scribes could also train other scribes; but the only two OB scribal teachers whom we know at all intimately both worked primarily as temple administrators and did a little teaching on the side. Ku-Ningal served the moon-god Sîn in the southern city of Ur in the mid-18th century [Charpin 1986, 432], while Ur-Utu worked for the temple of the sun-god Shamash in Sippar, near modern Baghdad, towards the end of the 17th century [Gasche 1989, 40–41]. Clearly there were scribes whose job it was to compose new literary

[34] If one assumes that both halves of the reciprocal pair were allowed up to four places, then one needs to add a further three pairs (Table 8, lines 6a, 9a, 12a). But all result in two or more of the short side, diagonal, and long side having too many sexagesimal places.

[35] At the end of a careful study of the demography of Greek mathematicians, Netz [1999, 291] concludes that "in [Classical] antiquity, each year saw the birth of a single mathematician on average, perhaps less. A handful of people interested passively in mathematics may have been born as well, but not more than a handful and, possibly, their numbers were quite negligible. In every generation, then, a few dozens at most of active mathematicians had to discover each other and to reach for their tiny audience. . . . They were thus doubly isolated, in time and in space."

works in Sumerian, and later Akkadian: royal praise poetry was commissioned for almost all the major kings of the early second millennium.[36] But while there was a steady demand for Sumerian literature in both cult and court, in neither institution was there a market for new mathematics.

So we are left, as I mentioned at the beginning, with an educational setting for mathematical creativity: new problems and scenarios designed to develop the mathematical competence of trainee scribes. Despite the fact that very little OB mathematics is satisfactorily traceable to excavated schools, there is a good deal of supporting evidence contained in the artefacts themselves. One of the most obvious is they fall comfortably into an educational typology, broadly comprising teachers' output (e.g., textbooks, model solutions with instructions) and students' output (e.g., laboriously copied mathematical tables, worked solutions to problems) [Robson 1999, 174–179]. The purpose of the text books and model solutions was to teach appropriate methods for solving mathematical problems. The numerical values taken by the parameters in the instructions were purely illustrative and were chosen to produce simple answers and avoid tricky arithmetical procedures. Half a dozen or so mathematical tablets are known which appear to have been written by teachers in the process of finding sets of numerically "nice" problems to give to their class. Indeed, Friberg [1981b, 62] has demonstrated that the scribbled numbers written on the reverse of one of those lists were written in the course of determining appropriate values for the last set of problems in the list. These lists of first lines appear to have provided the teacher with numerically simple variants on a particular set of problems so that the students could each be given individual numerical practice in a procedure. Support for this hypothesis can be adduced from two sets of multiplication exercises from Ur. The product of three numbers (which are different on each tablet) is multiplied by 6 40 in one group of five tablets, in the other groups of five it is divided successively by 10 and 30. We might understand them as the calculation of volumes (where the length, width, and height are different in each case), which are multiplied or divided by specific coefficients [Robson 1999, 253–259].

It has already been suggested that

> the purpose of the author of Plimpton 322 was to write a "teacher's aid" for setting up and solving problems involving right triangles [Friberg 1981a, 302],

or, alternatively,

> that the Plimpton tablet has nothing to do with Pythagorean triplets or trigonometry but, instead, is a pedagogical tool intended to help a mathematics teacher of the period make up a large number of igi–igibi [i.e., reciprocal-pair] quadratic equation exercises having known solutions and intermediate solution steps that are easily checked. [Buck 1980, 344]

This is entirely in line with what is known of the educational milieu of OB mathematics [Robson 1999, 172–183], and would also explain why, as argued above, the lengths of the short sides and diagonals in Plimpton 322 were restricted to 2 1/2 sexagesimal places or fewer: longer numerical strings would lead to difficult calculations that could interfere with the students' learning of mathematical method.[37]

[36] Now published by Black *et al.* [1998–].

[37] Cf. Robson [1999, 9–10], where Plimpton 322 must now be added to the list of known OB "catalogues" or teachers' aids.

But we are still left with a decision to make: was this teachers' list a compendium of suitable sets of right triangles (Friberg) or reciprocal pairs (Voils/Buck)? If we go with Friberg, we have to explain why Column I is in the table and why there is no (extant) column for l. If on the other hand we follow Voils/Buck, the problem is to explain why Columns II and III contain values which have been scaled up by the factor l—and why the word "diagonal" crops up in the headings of Columns I and III. Neither Friberg [1981a, 300] nor Buck [1980, 344] can give a satisfactory explanation. Neither do they make this observation: that the method used to compile Plimpton 322 was in all likelihood something other than the mathematical problem-type that its compiler wanted to teach and test. For, as we have just seen [cf. Friberg 1981b], numerical examples for such problems were "cooked" so that the results were simple whole numbers. That "cooking" process (which presumably started from the results the students were supposed to obtain) must have been a more-or-less reverse procedure to the one intended for the students. Are the columns of Plimpton 322 arranged in cooking order or problem-solving order? What were the problems being cooked or solved? The answer to both questions should influence our attempts to restore the missing columns.

We can try to answer the first by making comparisons with other mathematical tablets. All six of the other teachers' lists we know of give the numerical parameters of the problem to be set, often embedded in a mathematical question, but the numerical answers are never given [Robson 1999, 9].

Calculations made in the course of finding the parameters were written roughly at the bottom of the tablet or do not survive at all—presumably because they were erased, or made on separate "rough work" tablets. Plimpton 322 does not simply contain starting parameters, as the headings make clear (below), but on the other hand, the orderliness and tidiness of the tablet (no scribbled numbers or erasures on the blank surfaces, for instance) and the copying errors speak against its being rough work.[38] These latter arguments, of tablet format, are stronger than the argument of content: we can easily imagine a teacher wanting to check the intermediate calculations and results. (The other extant teachers' lists are for relatively simple problems—the areas of squares and circles, for instance—and use one- or two-place starting parameters; the teacher would not presumably have needed to record the answers to such easy problems. The arithmetic of Plimpton 322, by contrast, is relatively involved.)

On balance, then, Plimpton 322 was probably (but not certainly!) a good copy of a teachers' list, with two or three columns, now missing, containing starting parameters for a set of problems, one or two columns with intermediate results (Column I and perhaps a missing column to its left), and two columns with final results (II–III). All that remains is for us to decide what the problem type might have been. Let us start by recapping the information in the headings of Plimpton 322. In Column I (with two, or at most three short columns missing before it) we have "the (area of the) holding-square of the diagonal, from which 1 is torn out so that the width comes up," and Columns II and III contain "the square-side of the width" and the "square-side of the diagonal," respectively. The heading of Column I and the error in line II 13 both reveal that II and III were calculated, directly or indirectly, from I. The presence of the holding-square indicates that cut-and-paste

[38] But the copying and calculation errors do not preclude its being a teacher's list: such mistakes are not, and have never been, the sole domain of the learner!

geometry is involved, whereas the occurrence of the diagonal strongly suggests that we are dealing with triangles too, or at least with configurations within which a diagonal can be located. Equally we need to remember that the width and diagonal of Column I are of a triangle of length 1 whereas those of Columns II–III are of a triangle of length l (which is different every time). Further, if we believe that Plimpton 322 was intended to be a list of parameters to aid the setting of school mathematics problems (and the typological evidence suggests that it was), the question "how was the tablet calculated?" does not have to have the same answer as the question "what problems does the tablet set?" The first can be answered most satisfactorily by reciprocal pairs, as first suggested half a century ago, and the second by some sort of right-triangle problems. That is perhaps as far as we can go on present evidence: without closer parallels we run the risk of crossing the fuzzy boundary from history to speculation. The Mystery of the Cuneiform Tablet has not yet been fully solved.

IN CONCLUSION: ON METHOD, MATERIAL, AND MATERIAL CULTURE

Parts of this article have been deliberately provocative and polemical. I have compared history of ancient mathematics as it is sometimes practiced to the workings of popular detective fiction, and, with Pingree, likened its treatment of ancient artefacts to "pearls in the dung heap." But I aim to provoke not defensive anger but rather some reflection on how we should be thinking and writing about the history of ancient mathematics. This discussion of Plimpton 322 has in part simply been a ruse to attract attention: I would be much less interested in this interpretation reaching the general histories than in seeing presentations there of a wider variety of Mesopotamian mathematics, approached in a more historically aware manner. It is said that there was very little self-consciousness in the ancient Near East [e.g., Larsen 1987, 224–225]; it can appear at times that there is no self-consciousness in our attitudes to its mathematics. Most particularly, we need to grasp the challenge of glimpsing what is often called the Big Picture: that is, to look beyond our currently favourite "texts" to begin exploring the mathematical environment and mindset of the ancient world and accept that it is disturbingly alien in character. While "it is a delusion to suppose that we could ever become contemporaries of the original readers" [Fowler 1985, 49], we need to be aware that

> Text allows for a range of different meanings, while at the same time restricting the possibilities as the reader is guided by the literary codes or instructions inherent in the text. . . . [Modern] readers' subjective response to alien, ancient literature will need to be moderated by a greater degree of (explicit) preparation for the encounter. [Black 1998, 48–49]

Mathematics is as textual as a poem or a bus ticket; at the same time mathematical artefacts have as much physicality as a pot sherd or a bus. As I hope I have demonstrated, linguistic analysis, text-critical approaches, historical sensitivity, and archaeological awareness can make significant contributions to the history of mathematics.

The English-speaking mathematical world has a picture of OB mathematics today which is based, I would guess, on one primary source—namely Neugebauer's *ESA* [1951]—with a little of his and Sach's *MCT* [1945] and Aaboe's *Episodes* [1964] thrown in. Some of the more adventurous general histories add in a little smattering of other publications from that same era, namely Baqir [1951] and Bruins & Rutten [1961]. But, although the *subject* of study is (with a roughly 10% margin of error) 4,000 years old, the *field* of study is relatively new—less than a century old. Over the past two decades the subject has sprung back into life, and the past decade in particular has witnessed a qualitative shift: scholars are no longer content to "domesticate" Mesopotamian mathematics into something resembling modern output. The classic volumes on which the general histories are based are now seriously out of date and limited in their subject matter. We should no longer be seduced into thinking that simple mathematics necessarily has a simple history. The ancient Near East—in particular the cuneiform-writing world—produced a vast quantity of high quality mathematics over a 3000-year period. We do ourselves (and those who wrote it) a huge disservice to restrict the history of ancient mathematics to a list of "clever" mathematical procedures, regurgitating the usual interpretations of the same tired tablets that appear in our books over and over again.

ACKNOWLEDGMENTS

I have many people and institutions to thank for their contributions to this article. I first read Buck [1980] as an undergraduate studying under David Fowler—who encouraged my entry into the field of cuneiform studies and has been a stimulating source of motivation ever since. The section on angle derives ultimately from my doctoral work (now Chapt. 3 of Robson [1999]) and can be found in preliminary form in Robson [1997]. The re-interpretation of Plimpton 322 was first aired on the math-history-list@enterprise.maa.org in the spring of 1996 [cf. Deakin 1996a; 1996b] and then presented in different drafts to the Portuguese History of Mathematics Association in November 1997; the mathematics faculty of the University of Groningen in January 1998; the Mathematical Association of America's Summer Institute in the History of Mathematics and its Use in Teaching (IHMT) in July 1999; and the Joint Mathematics Meeting in New Orleans in January 2001.[39] During its long period of gestation, I have been supported first by the University of Oxford, then from October 1997 to September 2000 by the British Academy, and since then by All Souls College. I heartily thank Henry Mendell for his marvellous Sexagesimal plug-in for MS Excel 4, without which this series of calculations would have been unbearably tedious. Jane Siegel of the Rare Books and Manuscripts Collection of Columbia University Libraries provided invaluable help in tracking down original documentation and permitted me to collate and copy the original tablet. Thanks too to Jeremy Black, Mike Deakin, Maria Fernanda Estrada, Esther Flückiger-Hawker, David Fowler, Jens Høyrup, Victor Katz, Jan van Maanen, Duncan Melville, Lynn Meskell, Karen Parshall, Carlos Correia de Sá, Jim Tattersall, Niek Veldhuis, and the IHMT "veterans" for listening to, reading, and commenting on the material in this article as it gradually took shape, and to Luke Treadwell for many pleasurable distractions as I was writing it.

REFERENCES

Aaboe, A. 1964. *Episodes from the early history of mathematics*, New Mathematical Library, Vol. 13. Washington, DC: Mathematical Association of America.

Al-Rawi, F. N. H., & Roaf, M. 1984. Ten Old Babylonian mathematical problems from Tell Haddad, Himrin. *Sumer* **43**, 175–218.

Bahrani, Z. 1998. Conjuring Mesopotamia: Imaginative geography and a world past. In *Archaeology under fire: Nationalism, politics and heritage in the eastern Mediterranean and Middle East*, L. Meskell, Ed., pp. 159–174. London/New York: Routledge.

Banks, E. J. no date. Description of four tablets sent to Plimpton. Unpublished manuscript, Columbia University Libraries Special Collections: Cuneiform Collection.

Banks, E. J., & Plimpton, G. A. 1922–23. Banks–Plimpton correspondence, Dec. 1922–Mar. 1923. Unpublished manuscript, Columbia University Libraries Special Collections: Cuneiform Collection.

Banks, E. J. 1923. Banks–Plimpton correspondence, May–Oct. 1923. Unpublished manuscript, Columbia University Libraries Special Collections: Cuneiform Collection.

Baqir, T. 1951. Some more mathematical texts from Tell Harmal. *Sumer* **7**, 28–45.

[39] The text of that talk, an abbreviated and simplified version of this paper, appears as Robson [2001].

Black, J. A. 1998. *Reading Sumerian poetry.* London: Athlone Press.

Black, J. A., Cunningham, G. G., Robson, E., & Zólyomi, G. G. 1998–. *The electronic text corpus of Sumerian literature,* <http://www-etcsl.orient.ox.ac.uk/>. Oxford: The Oriental Institute.

Bruins, E. M. 1949. On Plimpton 322. Pythagorean numbers in Babylonian mathematics. *Koninklijke Nederlandse Akademie van Wetenschappen. Proceedings* **52**, 629–632.

Bruins, E. M. 1954. Some mathematical texts. *Sumer* **10**, 55–61.

———1955. Pythagorean triads in Babylonian mathematics: The errors on Plimpton 322. *Sumer* **11**, 117–121.

———1967. Reciprocals and Pythagorean triads. *Physis* **9**, 373–392.

Bruins, E. M., & Rutten, M. 1961. *Textes mathématiques de Suse*, Mémoires de la Mission Archéologique en Iran, Vol. 34. Paris: Librairie Orientaliste Paul Geuthner.

Buck, R. C., 1980. Sherlock Holmes in Babylon. *American Mathematical Monthly* **87**, 335–345.

Calinger, R. 1999. *A contextual history of mathematics: To Euler.* Upper Saddle River, NJ: Prentice Hall.

Charpin, D. 1986. *Le clergé d'Ur au siècle d'Hammurabi (XIX^{e}–XVIII^{e} siècles av. J.-C.),* Hautes Études Orientales, Vol. 22. Paris: Librairie Droz.

Cooke, R. 1997. *The history of mathematics: A brief course.* New York: Wiley.

Deakin, M. 1996a. More on Babylon. *Function* **20**, 128–134.

———1996b. Yet more on Babylon. *Function* **20**, 163–165.

Donoghue, E. F. 1998. In search of mathematical treasures: David Eugene Smith and George Arthur Plimpton. *Historia Mathematica* **25**, 359–365.

Englund, E. K. 1991. Hard work—Where will it get you? Labor management in Ur III Mesopotamia. *Journal of Near Eastern Studies* **50**, 255–280.

Fowler, A. 1985. *Kinds of literature. An introduction to the theory of genres and modes.* Oxford: Clarendon Press.

Fowler, D. H., & Robson, E. 1998. Square root approximations in Old Babylonian mathematics: YBC 7289 in context. *Historia Mathematica* **25**, 366–378.

Friberg, J. 1981a. Methods and traditions of Babylonian mathematics: Plimpton 322, Pythagorean triples and the Babylonian triangle parameter equations. *Historia Mathematica* **8**, 277–318.

———1981b. Methods and traditions of Babylonian mathematics. II. An Old Babylonian catalogue text with equations for squares and circles. *Journal of Cuneiform Studies* **33**, 57–64.

———1987–1990. Mathematik. In *RealLexikon der Assyriologie*, Vol. 7, D. O. Edzard, Ed., pp. 531–585. Berlin/Leipzig: De Gruyter.

Gasche, H. 1989. *La Babylonie au 17^{e} siècle avant notre ére: Approche archéologique, problèmes et perspectives,* Mesopotamian History and Environment. Series II, Memoirs; Vol. 1. Ghent: University of Ghent.

Gillings, R. J. 1953. Unexplained errors in Babylonian cuneiform tablet Plimpton 322. *Australian Journal of Science* **16**, 54–56.

Høyrup, J. 1990a. Algebra and naive geometry. An investigation of some basic aspects of Old Babylonian mathematical thought. *Altorientalische Forschungen* **17**, 27–69, 262–354.

———1990b. Sub-scientific mathematics: Observations on a pre-modern phenomenon. *History of Science* **28**, 63–86.

———1996. Changing trends in the historiography of Mesopotamian mathematics: An insider's view. *History of Science* **34**, 1–32.

———1999a. Pythagorean "rule" and "theorem": Mirror of the relation between Babylonian and Greek mathematics. In *Babylon: Focus mesopotamischer Geschichte, Wiege früher Gelehrsamkeit, Mythos in der Moderne. 2. Internationales Colloquium der Deutschen Orient-Gesellschaft, 24.–26. März 1998 in Berlin*. Johannes Renger, Ed., pp. 393–407. Berlin: Deutsche Orient-Gesellschaft/Saarbrücken: SDV Saarbrücker Druckerei und Verlag.

———1999b. The finer structure of the Old Babylonian mathematical corpus. Elements of classification, with some results. In *Assyriologica et Semitica. Festschrift für Joachim Oelsner anläßlich seines 65. Geburtstages am 18. Februar 1997*, J. Marzahn & H. Neumann, Eds., Alter Orient und Altes Testament, Vol. 252, pp. 117–177. Kevelaer: Butzon & Berker/Neukirchen-Vluyn: Neukirchener Verlag.

Huber, P. 1957. Bemerkungen über mathematische Keilschrifttexte. *L'Enseignement Mathématique* **3**, 19–57.

Joyce, D. E. 1995. Plimpton 322, *available at* <http://aleph0.clarku.edu/∼djoyce/mathhist/plimpnote.html>. Department of Mathematics and Computer Science, Clark University.

Knuth, D. 1972. Ancient Babylonian algorithms. *Communications of the Association for Computing Machinery* **15**, 671–677.

———1976. Reciprocal table of Inakibit-Anu incorrectly rendered. *Communications of the Association for Computing Machinery* **19**, 108.

Kragh, H. 1987. *An introduction to the historiography of science.* Cambridge: Cambridge University Press.

Kuhrt, A. 1995. *The ancient Near East: C. 3000–300 BC,* Routledge History of the Ancient World, 2 vols. London: Routledge.

Larsen, M. T. 1987. The Mesopotamian lukewarm mind: Reflections on science, divination and literacy. In *Language, literature, and history: philological and historical studies presented to Erica Reiner*, F. Rochberg-Halton, Ed., American Oriental Series, Vol. 67, pp. 203–225. New Haven: American Oriental Society.

Mendelsohn, I. 1943. *Catalogue of the Babylonian tablets in the libraries of Columbia University,* Catalogue Series, Vol. 1. New York, NY: Columbia University Libraries.

Michalowski, P. 1995. Sailing to Babylon, reading the dark side of the moon. In *The study of the ancient Near East in the twenty-first century: The William Foxwell Albright Centennial Conference*, J. S. Cooper & G. M. Schwartz, Eds., pp. 177–193. Winona Lake, IN: Eisenbrauns.

Netz, R. 1999. *The shaping of deduction in Greek mathematics: A study in cognitive history*, Ideas in Context, Vol. 51. Cambridge: Cambridge University Press.

Neugebauer, O. 1935–1937. *Mathematische Keilschrift-Texte,* Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik, Vol. A3, 3 vols. Berlin: Verlag von Julius Springer.

———1951/1957/1969. *The exact sciences in antiquity.* Copenhagen: Munksgaard; 2nd ed./Princeton, NJ: Brown University Press; reprint ed./New York, NY: Dover. All page references are given according to the Dover edition.

Neugebauer, O., & Sachs, A. J. 1945. *Mathematical cuneiform texts*, American Oriental Series, Vol. 29. New Haven, CT: American Oriental Society and the American Schools of Oriental Research.

Nissen, H., Damerow, P., & Englund, R. K. 1993. *Archaic bookkeeping: Early writing and techniques of economic administration in the ancient Near East.* Chicago, IL/London: University of Chicago Press.

Pingree, D. 1992. Hellenophilia versus the history of science. *Isis* **83**, 554–563.

Plimpton, G. A. 1993. *A collector's recollections*, P. A. Plimpton, Ed. New York, NY: Columbia University Libraries.

Powell, M. A. 1976. The antecedents of Old Babylonian place notation and the early history of Babylonian mathematics. *Historia Mathematica* **3**, 417–439.

Price, D. J. de S. 1964. The Babylonian "Pythagorean triangle" tablet. *Centaurus* **10**, 1–13.

Roaf, M. 1990. *Cultural atlas of Mesopotamia and the ancient Near East.* Oxford: Facts on File.

Robson, E. 1997. Why are there 60 minutes in an hour and 360° in a circle? In *Actas do ProfMat 97,* A. Simões, Ed., pp. 67–74. Figuera da Foz, Portugal: ProfMat 97.

———1999. *Mesopotamian mathematics, 2100–1600 BC: Technical constants in bureaucracy and education,* Oxford Editions of Cuneiform Texts, Vol. 14. Oxford: Clarendon Press.

———2000a. Mathematical cuneiform tablets in Philadelphia. I. Problems and calculations. *SCIAMVS–Sources and Commentaries in Exact Sciences* **1**, 11–48.

———2000b. Mesopotamian mathematics: Some historical background. In *Using history to teach mathematics*, V. J. Katz, Ed., pp. 149–158. Washington, DC: Mathematical Association of America.

———2000c. The uses of mathematics in ancient Iraq, 6000–600 BC. In *Mathematics across cultures: The history of non-Western mathematics*, H. Selin, Ed., pp. 93–113. Dordrecht: Kluwer Academic Publishers.

———2000d. *Accounting for change: Problematising the development of tabular book-keeping in early Mesopotamia*. Paper given at the Workshop on the Social and Economic Implications of Accounting in the Ancient World, British Museum, 15–17 November 2000. To appear.

———2001. Words and pictures: New light on Plimpton 322. *American Mathematical Monthly*, to appear.

Sachs, A. J. 1947. Babylonian mathematical texts. I. Reciprocals of regular sexagesimal numbers. *Journal of Cuneiform Studies* **1**, 219–240.

Schmidt, O. 1980. On Plimpton 322. Pythagorean numbers in Babylonian mathematics. *Centaurus* **24**, 4–13.

Sigrist, M. 2001. *Andrews University Cuneiform Texts*, Vol. 5. Berrien Springs, MI: Andrews University Press.

Soubeyran, D. 1984. Textes mathématiques de Mari. *Revue d'Assyriologie et d'Archéologie Orientale* **78**, 19–48.

von Soden, W. 1959–1981. *Akkadisches Handwörterbuch*, 3 vols. Wiesbaden: Otto Harrassowitz.