Archimedes and the *Quadrature of the Parabola*

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MATH 110 – Topics in Mathematics
Mathematics Through Time

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Outline

Who was Archimedes?

- Lived ca. 287 - 212 BCE, mostly in Greek city of Syracuse in Sicily
- Studied many topics in what we would call mathematics, physics, engineering (less distinction between them at the time)
- We don't know much about his actual life; much of his later reputation was based on somewhat dubious anecdotes, e.g. the "eureka moment," inventions he was said to have produced to aid in defense of Syracuse during Roman siege in which he was killed, etc.
- Perhaps most telling: we do know he designed a tombstone for himself illustrating the discovery he wanted most to be remembered for (still existed in later Roman times – discussed by Plutarch, Cicero)

Figure: Sphere inscribed in cylinder of equal radius

$3V_{sphere} = 2V_{cyl}$ and $A_{sphere} = A_{cyl}$ (lateral area)

Recognized in antiquity as a master

"In weightiness of matter and elegance of style, no classical mathematics treatise surpasses the works of Archimedes. This was recognized in antiquity; thus Plutarch says of Archimedes' works:
'It is not possible to find in all geometry more difficult and intricate questions, or more simple and lucid explanations. Some ascribe this to his genius; while others think that incredible effort and toil produced these, to all appearances, easy and unlaboured results.' "

A. Aaboe, *Episodes from the early history of mathematics*

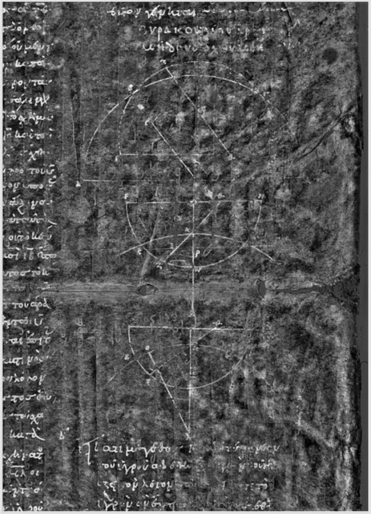
Surviving works

- *On the Equilibrium of Planes* (2 books)
- *On Floating Bodies* (2 books)
- *Measurement of a Circle*
- *On Conoids and Spheroids*
- *On Spirals*
- *On the Sphere and Cylinder* (2 books)

Surviving works, cont.

- *Book of Lemmas*, *The Sand-Reckoner*, *The Cattle-Problem*, the *Stomachion*
- *Quadrature of the Parabola*
- *The Method of Mechanical Theorems*
- *The Method* was long known only from references in other works
- 1906 – a *palimpsest* prayerbook created about 1229 CE (a reused manuscript) was found by to contain substantial portions (a 10th century CE copy from older sources)

A page from the palimpsest



Scientific context

- Archimedes flourished at height of Hellenistic period; Greek kingdoms (fragments of Alexander's empire) in Egypt, Syria strong patrons of science and mathematics, as was Syracuse
- State of the art in mathematics: 2nd or 3rd generation *after* Euclid, rough contemporary of Apollonius (best known for work on conic sections)
- Part of an active scientific community in fairly close communication; for instance several of the works cited above start with letters to colleagues
- Example: Letter to Eratosthenes (in Alexandria) at start of *The Method* (will discuss this later)

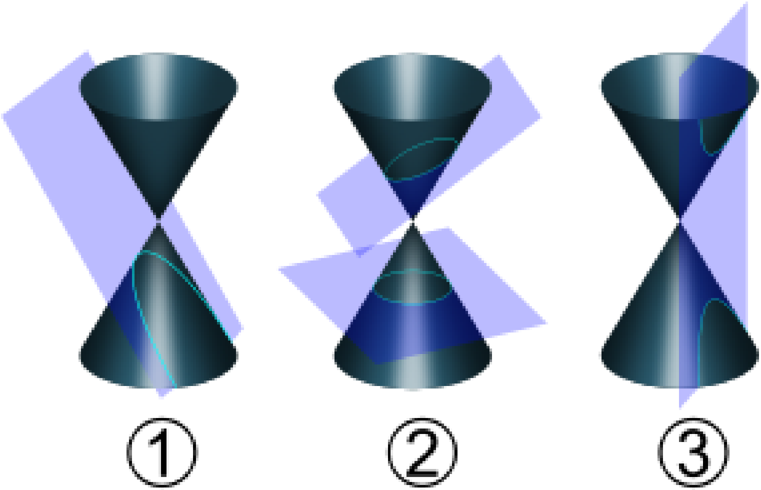
Goals for this class

- Discuss a problem treated by Archimedes in *The Method* and (later?) in *Quadrature of the Parabola* and the two different solutions Archimedes gave
- Think about the relation of the methods Archimedes developed and later innovations in mathematics such as limits, integrals, etc. some of you have studied if you have taken calculus
- Even if you have not seen some of this before, main goal is to appreciate *how far* Greek mathematics advanced after, and on the basis of Euclid's *Elements*

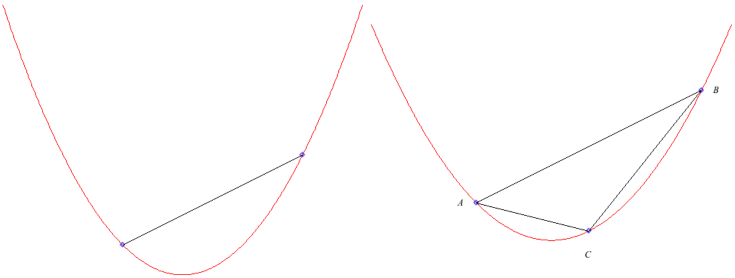
Conic Sections

- the curves obtained as plane slices of a cone:
- circles, ellipses, parabolas, hyperbolas (named by Apollonius of Perga, ca. 200 BCE)
- Euclid did not discuss these in the *Elements*
- But he wrote another text on *Conics* in 4 books, now lost. We know a fair amount about what it must have contained from references in other works (e.g. the texts of Archimedes that are the basis of what we will discuss today).

Conic Sections



Archimedes' problem: What is the area of a parabolic segment?



- Let C have same x-coordinate as the midpoint of AB
- The area of the parabolic segment $= \frac{4}{3}\mathrm{area}(\Delta ABC)$

"Quadrature of the Parabola"

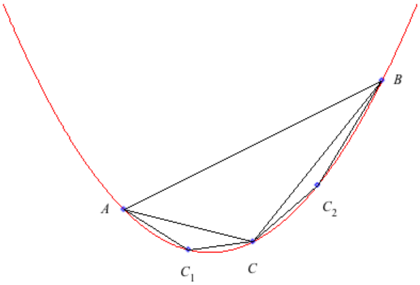
- In the work in the title of this slide, Archimedes gives two different arguments for this statement,
- One using the "method of exhaustion" – part of standard repertoire of Euclidean mathematics (laid out in Book 12 of *Elements*), based on work of Eudoxus
- A recurring theme in his mathematical work (used it as a standard method to obtain areas and volumes, approximate the value of π, etc.)
- One by the novel method also described in *The Method*
- We will look at both of these using modern notation – both quite interesting!

What's special about this C?

- Can show in several ways: The tangent to the parabola at C is *parallel* to AB
- So for this C, and this C only, the heights of the triangle and the parabolic segment are equal
- The point C is called the *vertex* of the segment

Sketch of Archimedes' first proof

- Now have two smaller parabolic segments; let their vertices be C_1 and C_2 and construct triangles ΔAC_1C and ΔCC_2B



The first proof, continued

- Archimedes shows that each of those triangles has area $\frac{1}{8}$ the area of ΔABC, so

$$\text{area}(\Delta AC_1C) + \text{area}(\Delta CC_2B) = \frac{1}{4}\text{area}(\Delta ABC).$$

- We do this construction some finite number n of times and add together the areas of all the triangles,

First proof, continued

- letting A_0 be the original area, $A_1 = \frac{1}{4}A_0$ the area of the first two smaller triangles, then $A_2 = \frac{1}{4}A_1 = \frac{1}{4^2}A_0$ the area of the next four, etc.
- we have total area

$$S_n = A_0 + \frac{1}{4}A_0 + \frac{1}{4^2}A_0 + \cdots + \frac{1}{4^n}A_0$$

- Moreover, $\frac{4}{3}A_0 - S_n = \frac{1}{3}A_n$, so as n increases without bound, S_n tends to $\frac{4}{3}A_0$

First proof, concluded

- On the other hand, as n increases without bound, the triangles constructed above *fill out* all of the area in the parabolic segment, so
- Archimedes concludes that the area of the parabolic segment $= \frac{4}{3}A_0 = \frac{4}{3}\mathrm{area}(\Delta ABC)$ as claimed before.
- We would derive all this, of course, using the usual properties of finite and infinite *geometric series*:

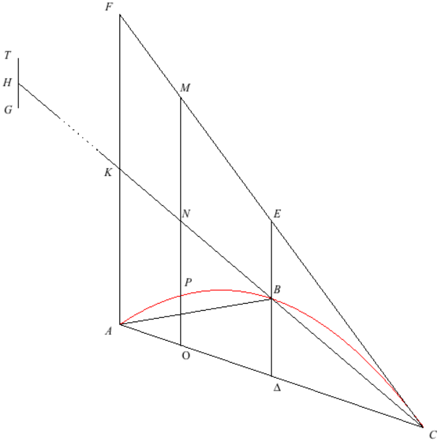
$$\text{area of segment} = \sum_{n=0}^{\infty} \frac{A_0}{4^n} = \frac{A_0}{1-\frac{1}{4}} = \frac{4A_0}{3}.$$

Archimedes had to do it all in an *ad hoc* way because that general theory did not exist yet!

Archimedes' second proof

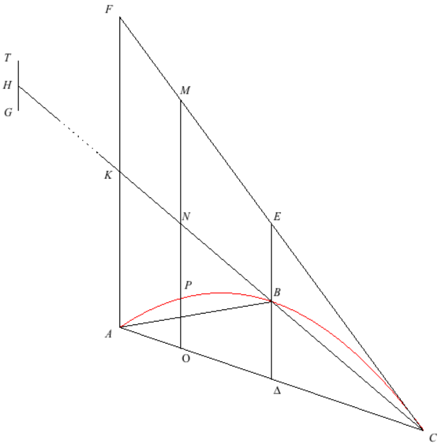
- Now, the remarkable second proof that Archimedes presents both in *Quadrature of the Parabola* and in *The Method*
- We'll follow the presentation in *The Method* (in Heath's translation) – this seems (to me) to be a less polished form of the argument and even more instructive about what Archimedes was getting at
- Key Idea: Apply a physical analogy – will compute the area of the parabolic segment essentially by considering it as a thin plate, or "lamina," of constant density and "weighing it on a balance" in an extremely clever way

The construction



- *Note:* Δ in figure $= D$ in text
- CF is tangent to the parabola at C
- B is the vertex
- O is an arbitrary point along the line from A to C
- ED, MO, FA are parallel to the axis of the parabola

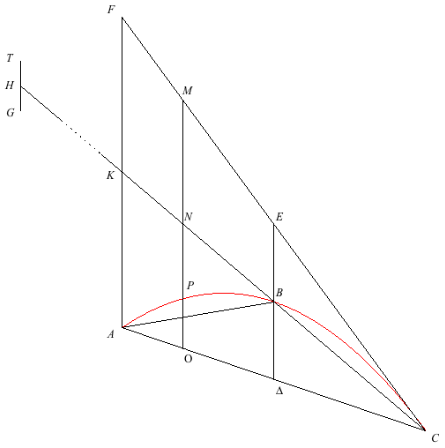
First observation



- By construction of the vertex B, $\text{area}(\Delta ADB) = \text{area}(\Delta CDB)$
- "Well-known" property of parabolas (at least in Archimedes' time!): $BE = BD$, $NM = NO$, $KA = KF$
- $\therefore$ $\text{area}(\Delta EDC) = \text{area}(\Delta ABC)$ and by similar triangles,

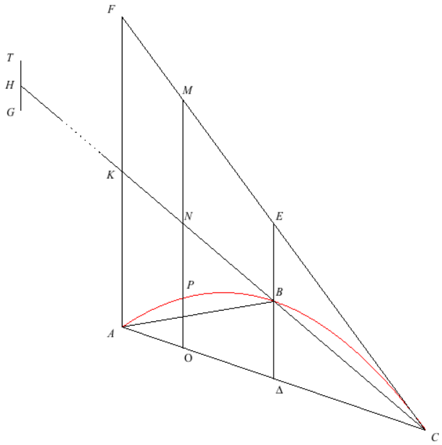
$$\text{area}(\Delta AFC) = 4 \cdot \text{area}(ABC)$$

Next steps



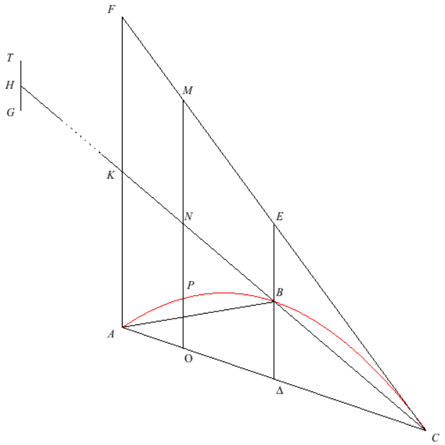
- Produce CK to H making $CK = KH$
- Consider CH "as the bar of a balance" with K at fulcrum
- Known facts about triangles (proved by Archimedes earlier): The *center of mass* ("balance point") of ΔAFC is located at a point W along CK with $CK = 3 \cdot KW$.

"Dissection" step



- Consider PO (think of a thin strip approximating the part of the parabolic segment with some thickness Δx)
- Properties of parabolas and similar triangles imply $MO : PO = AC : AO = CK : KN = HK : KN$
- Place segment $TG = PO$ with midpoint at H.
- Then Archimedes' *law of the lever* (fulcrum at K) says TG balances MO

Conclusion of the proof



- Do this for all "strips" like PO. We get that ΔACF exactly balances the whole collection of vertical strips making up the parabolic segment.
- By the law of the lever again, area(ΔACF) : area of segment $= HK : KW = 3 : 1$
- Since area(ΔACF) $=$ $4 \cdot$ area(ΔABC), the proof is complete.

From the introductory letter of *The Method* to Eratosthenes

"Seeing moreover in you, as I say, an earnest student, a man of considerable eminence in philosophy, and an admirer of mathematical inquiry, I thought fit to write out for you and explain in detail ... a certain method by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration."
Translation by T.L. Heath

What does this tell us?

- Can certainly say that Archimedes anticipated many ideas of *integral calculus* in this work
- But he was also *careful* and realized that he didn't have a complete justification for the idea of "balancing" an area with a collection of line segments
- So he also worked out the first "method of exhaustion" proof to supply an argument that would satisfy the mathematicians of his day

Archimedes' place in mathematical history

- Most historians of mathematics agree Archimedes was one of the greatest mathematicians in all of our known history – his work shows extremely strong originality, masterful use of an intuitive physical analogy, and technical power
- But he was so far ahead of his time that eventually very few, if any, people were reading him with complete understanding
- Almost miraculous that his works have survived
- Also extremely poignant to think what might have been if others had been better prepared to follow his lead at the time!