# Fans, Orbits, and Divisors on Toric Varieties
# Math in the Mountains Tutorial

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## Outline

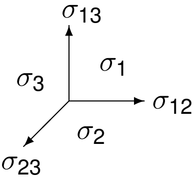
## Fans

### Definition

*A **fan** is a collection $\Sigma$ of SCRPC's in $N_{\mathbb{R}}$ such that*

- *If $\tau$ is a face of $\sigma \in \Sigma$, then $\tau \in \Sigma$*
- *if $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2 \in \Sigma$ and is a face of each.*

Example: Our picture from Hal's last talk:




shows a fan $\Sigma$ with three 2-dimensional cones $\sigma_1, \sigma_2, \sigma_3$, three 1-dimensional cones (the rays $\sigma_{ij} = \sigma_i \cap \sigma_j$), and one 0-dimensional cone $\{0\}$.

## The dual cones

We will also need to consider the duals of these cones in *M*:


Notice: $U_{\sigma_1} = \mathrm{Spec}\ \mathbb{C}[x, y]$ and $U_{\sigma_2} = \mathrm{Spec}\ \mathbb{C}[xy^{-1}, y^{-1}]$ Both are *contained in* $U_{\sigma_{12}} = \mathrm{Spec}\ \mathbb{C}[x, y, y^{-1}]$. By localization:

$$\mathbb{C}[x, y] \longrightarrow \mathbb{C}[x, y]_{\langle 1/y \rangle} = \mathbb{C}[x, y, y^{-1}]$$
$$\mathbb{C}[xy^{-1}, y^{-1}] \longrightarrow \mathbb{C}[xy^{-1}, y^{-1}]_{\langle 1/y^{-1} \rangle} = \mathbb{C}[x, y, y^{-1}]$$

(This is the reason for the slightly weird $\sigma^\vee$ construction: smaller faces $\leftrightarrow$ smaller affines.)

## Exercises

- Show that the above example gives a collection of affines that glue together to give $\mathbb{P}^2$
- Identify the cones and the gluing for the complete fans with 1-dimensional cones generated by
  1. $\pm e_1, \pm e_2$ (four 2-dimensional cones, four 1-dimensional cones, one 0-dimensional cone).
  2. $e_1, e_2, -e_2, -e_1 + ae_2$, where $a$ is any integer $\geq 1$ (four 2-dimensional cones, four 1-dimensional cones, one 0-dimensional cone).

**Abstract toric varieties $X_\Sigma$**

As in these examples, any fan $\Sigma$ gives the combinatorial information to produce affine torics gluing together to form an *abstract toric variety* $X_\Sigma$.

Most work on toric varieties in algebraic geometry uses this definition.

All $X_\Sigma$ are *normal* varieties (all affines $U_\sigma$ come from saturated semigroups so coordinate rings are integrally closed). So not as general as some examples $Y_\mathcal{A}$ we saw in first talks.

## Theorem (Orbit-cone correspondence)

*In the toric variety $X_\Sigma$, there is a 1-1 correspondence*

$$\{\textit{cones } \sigma \in \Sigma\} \leftrightarrow \{T_N - \textit{orbits in } X_\Sigma\}$$
$$\sigma \leftrightarrow O(\sigma) = \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*).$$

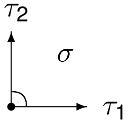
*Under this correspondence,*

1. $\dim O(\sigma) = \dim N - \dim \sigma$
2. $U_\sigma = \cup_{\tau \text{ face of } \sigma} O(\tau)$.
3. *If $\tau$ is a face of $\sigma$, then $O(\sigma) \subset \overline{O(\tau)}$ and* $\overline{O(\tau)} = \cup_{\tau \text{ face of } \sigma} O(\sigma)$.

Key technical point: Uses identification $p \in U_\sigma$ as semigroup homomorphisms $S_\sigma \to \mathbb{C}$ via $p \mapsto (m \mapsto \chi^m(p))$.

## If that didn't make sense, try this example, ...

Let $\Sigma$ be the following (non-complete) fan in $N = \mathbb{R}^2$:




The corresponding toric variety $X_\Sigma$ is $\mathbb{C}^2$; covering by affines

$$
\begin{aligned}
U_\sigma &= \mathrm{Spec}\ \mathbb{C}[x, y] = \mathbb{C}^2 \\
U_{\tau_1} &= \mathrm{Spec}\ \mathbb{C}[x, y, y^{-1}] = \mathbb{C} \times \mathbb{C}^* \\
U_{\tau_2} &= \mathrm{Spec}\ \mathbb{C}[x, x^{-1}, y] = \mathbb{C}^* \times \mathbb{C} \\
U_{\{0\}} &= \mathrm{Spec}\ \mathbb{C}[x, x^{-1}, y, y^{-1}] = \mathbb{C}^* \times \mathbb{C}^*
\end{aligned}
$$

## The $(\mathbb{C}^*)^2$-orbits and how they correspond with the cones

| Cone | Orbit | Dimension |
|---|---|---|
| $\sigma$ | $\{(0,0)\}$ | 0 |
| $\tau_1$ | $\mathbb{C}^* \times \{0\}$ | 1 |
| $\tau_2$ | $\{0\} \times \mathbb{C}^*$ | 1 |
| $\{0\}$ | $(\mathbb{C}^*)^2$ | 2 |

Note: For instance for the cone $\tau_1$, $\tau_1^\perp \cap M \simeq \mathbb{Z}$ is the subgroup generated by $e_2$. For the 0-dimensional cone $\{0\}$, we have $\{0\}^\perp \cap M \simeq \mathbb{Z}^2$.

Check:

$$U_{\tau_1} = O(\tau_1) \cup O(\{0\}) = (\mathbb{C}^* \times \{0\}) \cup (\mathbb{C}^* \times \mathbb{C}^*) = \mathbb{C}^* \times \mathbb{C}$$

## Toric morphisms, II

If $\Sigma_1 \subset N_1$ and $\Sigma_2 \subset N_2$ are fans, then we say a $\mathbb{Z}$-linear $\varphi : N_1 \to N_2$ is *compatible* with the fans if it maps cones to cones:

$$\sigma_1 \in \Sigma_1 \Rightarrow \varphi(\sigma_1) \subset \sigma_2 \in \Sigma_2$$

(for some $\sigma_2$).

**Theorem**

*"Toric morphisms are the same as in the affine case" – that is, any toric morphism between $X_{\Sigma_1}$ and $X_{\Sigma_2}$ is given by a mapping of this form.*

## Weil Divisors

**Definition**

*Let $X$ be an irreducible variety. $\mathrm{Div}(X)$ is the free abelian group generated by the irreducible codimension-1 subvarieties of $X$. The elements of $\mathrm{Div}(X)$ are called* ***Weil divisors*** *on $X$.*

Key: If $R$ is a normal domain and $P$ is a codimension-1 prime ideal of $R$, then the localization $R_P$ is a DVR. This gives a way to talk about orders of zeros and poles along $V(P)$.

Example: Let $P = \langle x \rangle$ in $\mathbb{C}[x, y]$. If $f = \alpha/\beta \in \mathbb{C}(x, y)$, can find $n_i \in \mathbb{Z}$ such that $x^{n_1} | \alpha$ and $x^{n_1} | \beta$ (maximal). Then $\mathrm{ord}_{\langle x \rangle}(f) = n_1 - n_2$.

## Divisors on curves

- The following special case may be familiar: Say $X$ is a smooth, compact curve over $\mathbb{C}$, $=$ compact Riemann surface
- There are no nonconstant holomorphic functions on $X$
- But can consider the meromorphic functions on $X$ with poles bounded by a divisor $D = \sum a_i p_i$, $a_i \in \mathbb{N}$: $H^0(D) = \{f \mid (f) + D \geq 0\}$, a finite-dimensional $\mathbb{C}$-vector space
- For any holomorphic 1-form $\omega$ on $X$, $\sum_{p \in X} \mathrm{res}_p(f\omega) = 0$. Analysis of local Laurent series of $f$'s in $H^0(D)$ gives Riemann's inequality: $\dim H^0(D) \geq \deg(D) + 1 - g(X)$, and Riemann-Roch identifies the difference between left and right sides.

## The divisor class group

**Definition**

*The divisor class group is*

$$\mathrm{Cl}(X) = \mathrm{Div}(X)/\sim,$$

*where $D_1 \sim D_2$ if $D_1 - D_2 = (f)$ for some $f \in \mathbb{C}(X)$ ("linear equivalence").*

Exercise: $\mathrm{Cl}(\mathbb{P}^1) = \mathbb{Z}$. (Hartshorne says: "$\mathrm{Cl}(X)$ is a very important invariant, in general not easy to calculate." This case is feasible, though!)

## $\mathrm{Cl}(X_\Sigma)$ for a toric variety

**Theorem**

*There is an exact sequence*

$$M \xrightarrow{A} \mathbb{Z}^{|\Sigma(1)|} \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0$$

*where $A$ is the matrix whose rows are the primitive lattice generators of the* 1*-dimensional cones of $\Sigma$.*

Sketch of proof: First $\mathbb{Z}^{|\Sigma(1)|} \to \mathrm{Cl}(X_\Sigma) \to \mathrm{Cl}(T_N) \to 0$ is exact since $X_\Sigma$ is the union of the "big torus" isomorphic to $T_N$ and the closures of the orbits corresponding to the 1-dimensional cones. Since $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a UFD, $\mathrm{Cl}(T_N) = \{0\}$. For any $m \in M$, $(\chi^m) = 0$ in $\mathrm{Cl}(X_\Sigma)$. Hence $M \subseteq \ker(\mathbb{Z}^{|\Sigma(1)|} \to \mathrm{Cl}(X_\Sigma))$. Exercise: finish this proof.

## An example

- For $X_\Sigma = \mathbb{P}^2$ from the fan $\Sigma$ considered earlier we have

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z}^3 \longrightarrow \mathrm{Cl}(\mathbb{P}^2) \longrightarrow 0$$

  Hence $\mathrm{Cl}(\mathbb{P}^1) \simeq \mathbb{Z}$.
- Similarly, we have $\mathrm{Cl}(\mathbb{P}^n) \simeq \mathbb{Z}$ for all $n \geq 1$.

## Another example

- For $X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$ from the fan $\Sigma$ with 1-dimensional cones spanned by $\pm e_1, \pm e_2$ we have

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}} \mathbb{Z}^4 \longrightarrow \mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \longrightarrow 0$$

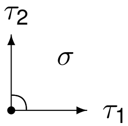
  Hence $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z}^2$.
- So now, when you see $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)$, you know what it means(!) (See Hartshorne, V, III.)

## Divisors

**Theorem**

*Let $\rho \in \Sigma(1)$, and $D_\rho$ the corresponding divisor. Then $v_{D_\rho}(\chi^m) = \langle m, u_\rho \rangle$, where $u_\rho$ is the primitive lattice vector in that cone (the "first lattice point" on the ray).*

Example: For $\Sigma$ below




take $\tau_1 = \mathrm{cone}(e_1)$. Then $v_{\tau_1}(x^2y) = \langle (2,1), (1,0) \rangle = 2$.

## Divisors give maps

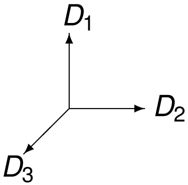
- Also *Cartier* = "locally principal" divisors. It is these that correspond to invertible sheaves = line bundles. If $X_\Sigma$ is smooth, though, the two notions *coincide*, so we'll blur the distinction(!)
- Given $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$, a basis of $H^0(\mathcal{O}_{X_\Sigma}(D))$ can be used to define a map $X_\Sigma \to \mathbb{P}^n$ for $n = \dim H^0(D) + 1$.
- We can ask whether the map is an isomorphism.
- We investigate methods for finding a basis of $H^0(\mathcal{O}_{X_\Sigma}(D))$ next.

## The global sections

**Theorem**

$$H^0\left(\mathcal{O}_{X_\Sigma}\left(\sum_\rho a_\rho D_\rho\right)\right) = \mathrm{Span}\{\chi^m \mid \langle m, u_\rho\rangle \geq -a_\rho \textit{ for all } \rho\}.$$

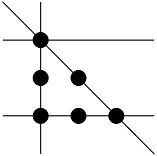
Example: For $\mathbb{P}^2$ with $\Sigma$:




Let $D = 2D_1$. $H^0(2D_1)$ is spanned by the $\chi^m$ for $m = (a, b)$ satisfying $\langle (a, b), (0, 1)\rangle \geq -2$, $\langle (a, b), (1, 0)\rangle \geq 0$ and $\langle (a, b), (-1, -1)\rangle \geq 0$.

**Example, continued**

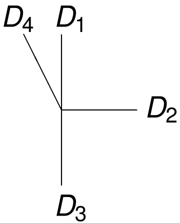
The inequalities $b \geq -2$, $a \geq 0$ and $a + b \leq 0$ define a polytope as follows



The lattice points are a translate of the lattice points in $\mathrm{conv}\{(0,0),(2,0),(0,2)\}$. Hence we get a map to $\mathbb{P}^5$ whose image is the degree 2 Veronese image of $\mathbb{P}^2$ (same as that from $\mathcal{O}_{\mathbb{P}^2}(2H)$).

## An exercise

For $\Sigma$ as below




with the ray corresponding to $D_4$ equal to $\mathrm{cone}(-e_1 + 2e_2)$. Compute $H^0(D_1 + D_2)$.

## Support functions, ampleness

### Definition

*A **support function** for $D = \sum_\rho a_\rho D_\rho$ is a function $\varphi_D : |\Sigma| \to \mathbb{R}$ such that*

- *$\varphi_D$ is linear on each cone in $\Sigma$*
- *$\varphi_D(u_\rho) = -a_\rho$ for all $\rho \in \Sigma(1)$*

### Theorem

- *If $\Sigma$ has convex support, and is full-dimensional, then $D$ is basepoint-free $\Leftrightarrow$ $\varphi_D$ is convex*
- *If $\Sigma$ is complete, then $D$ is ample $\Leftrightarrow$ $\varphi_D$ is strictly convex*

**An Example, Demazure's theorem**

Consider $2D_1$ on $\mathbb{P}^2$.

[Draw graph of a support function on board(!)]

**Theorem (Demazure)**

*If $\Sigma$ has convex support and $D$ is basepoint-free, the $H^i(D) = 0$ for $i > 0$.*

## The quotient construction

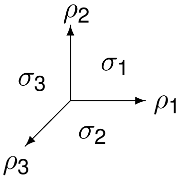
- Instead of patching affines to get $\mathbb{P}^n$, recall can build this toric variety globally as $(\mathbb{C}^{n+1} \setminus \{0\})/\sim$
- Cox: There is an analog of this for any toric variety(!)
- Define $\mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ and an ideal

$$B_\Sigma = \langle x^{\widehat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho \mid \sigma \in \Sigma \rangle$$

- Analog of $\mathbb{C}^{n+1} \setminus \{0\}$ is $\mathbb{C}^{|\Sigma(1)|} \setminus V(B_\Sigma)$

## Sanity check

- For the standard fan $\Sigma$ defining $\mathbb{P}^2$, labeled like this:




we have $x^{\widehat{\sigma_1}} = x_3$, $x^{\widehat{\sigma_2}} = x_2$, $x^{\widehat{\sigma_3}} = x_1$. So $V(B_\Sigma) = V(x_1, x_2, x_3)$ as expected.

## The quotient map

- If $N_{\mathbb{R}}$ is spanned by the $u_\rho$ for $\rho \in \Sigma(1)$, then

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{|\Sigma(1)|} \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0$$

  is exact.
- Hit this with $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ get

$$0 \longrightarrow G \longrightarrow (\mathbb{C}^*)^{|\Sigma(1)|} \longrightarrow T_N$$

- Since $G \subseteq (\mathbb{C}^*)^{|\Sigma(1)|}$, $G$ acts on $\mathbb{C}^{|\Sigma(1)|}$
- Modulo some (hard) technical details about quotients by algebraic group actions (GIT) we get $X_\Sigma$ as the quotient modulo this action

## Sanity check, II

For $\mathbb{P}^2$, the first exact sequence above is

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}} \longrightarrow \mathrm{Cl}(\mathbb{P}^2) \longrightarrow 0$$

After taking $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$, we have

$$(\mathbb{C}^*)^2 \xleftarrow{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}} (\mathbb{C}^*)^3 \xleftarrow{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \mathbb{C}^* \longleftarrow 0$$

And indeed $\mathbb{C}^*$ acts on $\mathbb{C}^3$ via $a \sim \lambda a$.

## Why is this important?

### Theorem

*Any coherent sheaf on a toric variety $X_\Sigma$ comes from a finitely-generated $\mathrm{Cl}(X_\Sigma)$-graded module over the "Cox ring"*

$$S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$$

(Generalizes a result of Serre in the case of $X_\Sigma = \mathbb{P}^n$, where the "Cox ring" is the same as the homogeneous coordinate ring $\mathbb{C}[x_0, \ldots, x_n]$.)

Translates abstract questions about sheaf cohomology into *concrete, computable* questions in multigraded commutative algebra(!)