Fans, Orbits, and Divisors on Toric Varieties
Math in the Mountains Tutorial

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Outline

1. Fans, cones, and orbits
2. Divisors
3. Toric varieties via quotients
Fans

Definition

A fan is a collection $\Sigma$ of SCRPC's in $N_\mathbb{R}$ such that

- If $\tau$ is a face of $\sigma \in \Sigma$, then $\tau \in \Sigma$
- if $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2 \in \Sigma$ and is a face of each.

Example: Our picture from Hal’s last talk:

shows a fan $\Sigma$ with three 2-dimensional cones $\sigma_1, \sigma_2, \sigma_3$, three 1-dimensional cones (the rays $\sigma_{ij} = \sigma_i \cap \sigma_j$), and one 0-dimensional cone $\{0\}$. 
The dual cones

We will also need to consider the duals of these cones in $M$:

\[
\sigma_3^\vee \quad \sigma_1^\vee \quad \sigma_2^\vee \quad \sigma_{12}^\vee
\]

Notice: $U_{\sigma_1} = \text{Spec } \mathbb{C}[x, y]$ and $U_{\sigma_2} = \text{Spec } \mathbb{C}[xy^{-1}, y^{-1}]$ Both are contained in $U_{\sigma_{12}} = \text{Spec } \mathbb{C}[x, y, y^{-1}]$. By localization:

\[
\mathbb{C}[x, y] \longrightarrow \mathbb{C}[x, y]_{1/y} = \mathbb{C}[x, y, y^{-1}]
\]
\[
\mathbb{C}[xy^{-1}, y^{-1}] \longrightarrow \mathbb{C}[xy^{-1}, y^{-1}]_{1/y^{-1}} = \mathbb{C}[x, y, y^{-1}]
\]

(This is the reason for the slightly weird $\sigma^\vee$ construction: smaller faces $\leftrightarrow$ smaller affines.)
Exercises

- Show that the above example gives a collection of affines that glue together to give $\mathbb{P}^2$.

- Identify the cones and the gluing for the complete fans with 1-dimensional cones generated by
  1. $\pm e_1, \pm e_2$ (four 2-dimensional cones, four 1-dimensional cones, one 0-dimensional cone).
  2. $e_1, e_2, -e_2, -e_1 + ae_2$, where $a$ is any integer $\geq 1$ (four 2-dimensional cones, four 1-dimensional cones, one 0-dimensional cone).
Abstract toric varieties $X_{\Sigma}$

As in these examples, any fan $\Sigma$ gives the combinatorial information to produce affine torics gluing together to form an abstract toric variety $X_{\Sigma}$.

Most work on toric varieties in algebraic geometry uses this definition.

All $X_{\Sigma}$ are normal varieties (all affines $U_{\sigma}$ come from saturated semigroups so coordinate rings are integrally closed). So not as general as some examples $Y_\mathcal{A}$ we saw in first talks.
Theorem (Orbit-cone correspondence)

In the toric variety $X_\Sigma$, there is a 1-1 correspondence

\[
\{\text{cones } \sigma \in \Sigma\} \leftrightarrow \{T_N - \text{orbits in } X_\Sigma\}
\]

\[
\sigma \leftrightarrow O(\sigma) = \text{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*)
\]

Under this correspondence,

1. $\dim O(\sigma) = \dim N - \dim \sigma$
2. $U_\sigma = \bigcup_{\tau \text{ face of } \sigma} O(\tau)$.
3. If $\tau$ is a face of $\sigma$, then $O(\sigma) \subset O(\tau)$ and $\overline{O(\tau)} = \bigcup_{\tau \text{ face of } \sigma} O(\sigma)$.

Key technical point: Uses identification $p \in U_\sigma$ as semigroup homomorphisms $S_\sigma \to \mathbb{C}$ via $p \mapsto (m \mapsto \chi^m(p))$. 
If that didn’t make sense, try this example, ...

Let $\Sigma$ be the following (non-complete) fan in $N = \mathbb{R}^2$:

$$\tau_2$$

$$\sigma$$

$$\tau_1$$

The corresponding toric variety $X_\Sigma$ is $\mathbb{C}^2$; covering by affines

$$U_\sigma = \text{Spec } \mathbb{C}[x, y] = \mathbb{C}^2$$

$$U_{\tau_1} = \text{Spec } \mathbb{C}[x, y, y^{-1}] = \mathbb{C} \times \mathbb{C}^*$$

$$U_{\tau_2} = \text{Spec } \mathbb{C}[x, x^{-1}, y] = \mathbb{C}^* \times \mathbb{C}$$

$$U_{\{0\}} = \text{Spec } \mathbb{C}[x, x^{-1}, y, y^{-1}] = \mathbb{C}^* \times \mathbb{C}^*$$
The $(\mathbb{C}^*)^2$-orbits and how they correspond with the cones

<table>
<thead>
<tr>
<th>Cone</th>
<th>Orbit</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>${(0, 0)}$</td>
<td>0</td>
</tr>
<tr>
<td>$\tau_1$</td>
<td>$\mathbb{C}^* \times {0}$</td>
<td>1</td>
</tr>
<tr>
<td>$\tau_2$</td>
<td>${0} \times \mathbb{C}^*$</td>
<td>1</td>
</tr>
<tr>
<td>${0}$</td>
<td>$(\mathbb{C}^*)^2$</td>
<td>2</td>
</tr>
</tbody>
</table>

Note: For instance for the cone $\tau_1$, $\tau_1^\perp \cap M \simeq \mathbb{Z}$ is the subgroup generated by $e_2$. For the 0-dimensional cone $\{0\}$, we have $\{0\}^\perp \cap M \simeq \mathbb{Z}^2$.

Check:

$$U_{\tau_1} = O(\tau_1) \cup O(\{0\}) = (\mathbb{C}^* \times \{0\}) \cup (\mathbb{C}^* \times \mathbb{C}^*) = \mathbb{C}^* \times \mathbb{C}$$
Toric morphisms, II

If $\Sigma_1 \subset N_1$ and $\Sigma_2 \subset N_2$ are fans, then we say a $\mathbb{Z}$-linear $\varphi : N_1 \rightarrow N_2$ is compatible with the fans if it maps cones to cones:

$$\sigma_1 \in \Sigma_1 \Rightarrow \varphi(\sigma_1) \subset \sigma_2 \in \Sigma_2$$

(for some $\sigma_2$).

Theorem

“Toric morphisms are the same as in the affine case” – that is, any toric morphism between $X_{\Sigma_1}$ and $X_{\Sigma_2}$ is given by a mapping of this form.
Weil Divisors

**Definition**

Let $X$ be an irreducible variety. $\text{Div}(X)$ is the free abelian group generated by the irreducible codimension-1 subvarieties of $X$. The elements of $\text{Div}(X)$ are called **Weil divisors** on $X$.

Key: If $R$ is a normal domain and $P$ is a codimension-1 prime ideal of $R$, then the localization $R_P$ is a DVR. This gives a way to talk about orders of zeros and poles along $V(P)$.

Example: Let $P = \langle x \rangle$ in $\mathbb{C}[x, y]$. If $f = \alpha/\beta \in \mathbb{C}(x, y)$, can find $n_i \in \mathbb{Z}$ such that $x^{n_1}|\alpha$ and $x^{n_1}|\beta$ (maximal). Then $\text{ord}_{\langle x \rangle}(f) = n_1 - n_2$. 
Divisors on curves

The following special case may be familiar: Say $X$ is a smooth, compact curve over $\mathbb{C}$, = compact Riemann surface

There are no nonconstant holomorphic functions on $X$

But can consider the meromorphic functions on $X$ with poles bounded by a divisor $D = \sum a_i p_i$, $a_i \in \mathbb{N}$: $H^0(D) = \{ f \mid (f) + D \geq 0 \}$, a finite-dimensional $\mathbb{C}$-vector space

For any holomorphic 1-form $\omega$ on $X$, $\sum_{p \in X} \text{res}_p(f \omega) = 0$. Analysis of local Laurent series of $f$'s in $H^0(D)$ gives Riemann’s inequality: $\dim H^0(D) \geq \deg(D) + 1 - g(X)$, and Riemann-Roch identifies the difference between left and right sides.
The divisor class group

Definition

The divisor class group is

\[ \text{Cl}(X) = \text{Div}(X) / \sim, \]

where \( D_1 \sim D_2 \) if \( D_1 - D_2 = (f) \) for some \( f \in \mathbb{C}(X) \) ("linear equivalence").

Exercise: \( \text{Cl}(\mathbb{P}^1) = \mathbb{Z} \). (Hartshorne says: "\( \text{Cl}(X) \) is a very important invariant, in general not easy to calculate." This case is feasible, though!)
\( \text{Cl}(X_\Sigma) \) for a toric variety

**Theorem**

There is an exact sequence

\[
M \xrightarrow{A} \mathbb{Z}^{|\Sigma(1)|} \longrightarrow \text{Cl}(X_\Sigma) \longrightarrow 0
\]

where \( A \) is the matrix whose rows are the primitive lattice generators of the 1-dimensional cones of \( \Sigma \).

Sketch of proof: First \( \mathbb{Z}^{|\Sigma(1)|} \longrightarrow \text{Cl}(X_\Sigma) \longrightarrow \text{Cl}(T_N) \longrightarrow 0 \) is exact since \( X_\Sigma \) is the union of the “big torus” isomorphic to \( T_N \) and the closures of the orbits corresponding to the 1-dimensional cones. Since \( \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) is a UFD, \( \text{Cl}(T_N) = \{0\} \). For any \( m \in M \), \( (\chi^m) = 0 \) in \( \text{Cl}(X_\Sigma) \). Hence \( M \subseteq \text{ker}(\mathbb{Z}^{|\Sigma(1)|} \longrightarrow \text{Cl}(X_\Sigma)) \).

Exercise: finish this proof.
An example

For $X_{\Sigma} = \mathbb{P}^2$ from the fan $\Sigma$ considered earlier we have

$$
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
-1 & -1
\end{pmatrix}
$$

$$
\mathbb{Z}^2 \rightarrow \mathbb{Z}^3 \rightarrow \text{Cl}(\mathbb{P}^2) \rightarrow 0
$$

Hence $\text{Cl}(\mathbb{P}^1) \cong \mathbb{Z}$.

Similarly, we have $\text{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$ for all $n \geq 1$. 
Another example

For $X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$ from the fan $\Sigma$ with 1-dimensional cones spanned by $\pm e_1, \pm e_2$ we have

$$
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
-1 & 0 \\
0 & -1
\end{pmatrix}
$$

$\mathbb{Z}^2 \to \mathbb{Z}^4 \to \text{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \to 0$

Hence $\text{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}^2$.

So now, when you see $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)$, you know what it means(!) (See Hartshorne, V, III.)
Divisors

Theorem

Let \( \rho \in \Sigma(1) \), and \( D_\rho \) the corresponding divisor. Then
\[ \nu_{D_\rho}(x^m) = \langle m, u_\rho \rangle, \]
where \( u_\rho \) is the primitive lattice vector in that cone (the “first lattice point” on the ray).

Example: For \( \Sigma \) below

\[ \sigma \]

\[ \tau_1 \]

\[ \tau_2 \]

\[ e_1 \]

Take \( \tau_1 = \text{cone}(e_1) \). Then \( \nu_{\tau_1}(x^2y) = \langle (2, 1), (1, 0) \rangle = 2 \).
Divisors give maps

- Also *Cartier* = “locally principal” divisors. It is these that correspond to invertible sheaves = line bundles. If $X_\Sigma$ is smooth, though, the two notions *coincide*, so we’ll blur the distinction(!)

- Given $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$, a basis of $H^0(\mathcal{O}_{X_\Sigma}(D))$ can be used to define a map $X_\Sigma \to \mathbb{P}^n$ for $n = \dim H^0(D) + 1$.

- We can ask whether the map is an isomorphism.

- We investigate methods for finding a basis of $H^0(\mathcal{O}_{X_\Sigma}(D))$ next.
The global sections

**Theorem**

\[
H^0 \left( \mathcal{O}_{\Sigma} \left( \sum_{\rho} a_{\rho} D_{\rho} \right) \right) = \text{Span}\{\chi^m | \langle m, u_{\rho} \rangle \geq -a_{\rho} \text{ for all } \rho \}\.
\]

Example: For \( \mathbb{P}^2 \) with \( \Sigma \):

Let \( D = 2D_1 \). \( H^0(2D_1) \) is spanned by the \( \chi^m \) for \( m = (a, b) \) satisfying \( \langle (a, b), (0, 1) \rangle \geq -2, \langle (a, b), (1, 0) \rangle \geq 0 \) and \( \langle (a, b), (-1, -1) \rangle \geq 0 \).
Example, continued
The inequalities \( b \geq -2, \ a \geq 0 \) and \( a + b \leq 0 \) define a polytope as follows

The lattice points are a translate of the lattice points in \( \text{conv}\{(0, 0), (2, 0), (0, 2)\} \). Hence we get a map to \( \mathbb{P}^5 \) whose image is the degree 2 Veronese image of \( \mathbb{P}^2 \) (same as that from \( \mathcal{O}_{\mathbb{P}^2}(2H) \)).
An exercise

For $\Sigma$ as below

with the ray corresponding to $D_4$ equal to cone($-e_1 + 2e_2$). Compute $H^0(D_1 + D_2)$. 
Support functions, ampleness

**Definition**

A **support function** for $D = \sum_{\rho} a_{\rho} D_{\rho}$ is a function $\varphi_D : |\Sigma| \to \mathbb{R}$ such that

- $\varphi_D$ is linear on each cone in $\Sigma$
- $\varphi_D(u_{\rho}) = -a_{\rho}$ for all $\rho \in \Sigma(1)$

**Theorem**

- If $\Sigma$ has convex support, and is full-dimensional, then $D$ is basepoint-free $\iff \varphi_D$ is convex
- If $\Sigma$ is complete, then $D$ is ample $\iff \varphi_D$ is strictly convex
An Example, Demazure’s theorem

Consider $2D_1$ on $\mathbb{P}^2$.

[Draw graph of a support function on board(!)]

**Theorem (Demazure)**

*If $\Sigma$ has convex support and $D$ is basepoint-free, the $H^i(D) = 0$ for $i > 0$.***
The quotient construction

- Instead of patching affines to get $\mathbb{P}^n$, recall can build this toric variety globally as $(\mathbb{C}^{n+1} \setminus \{0\}) / \sim$
- Cox: There is an analog of this for any toric variety(!)
- Define $\mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ and an ideal

$$B_\Sigma = \langle x^\hat{\sigma} = \prod_{\rho \notin \sigma(1)} x_\rho \mid \sigma \in \Sigma \rangle$$

- Analog of $\mathbb{C}^{n+1} \setminus \{0\}$ is $\mathbb{C}^{\vert \Sigma(1) \vert} \setminus V(B_\Sigma)$
Sanity check

- For the standard fan $\Sigma$ defining $\mathbb{P}^2$, labeled like this:

  \begin{align*}
  x^{\sigma_1} &= x_3, \\
  x^{\sigma_2} &= x_2, \\
  x^{\sigma_3} &= x_1.
  \end{align*}

  So $V(B_{\Sigma}) = V(x_1, x_2, x_3)$ as expected.
The quotient map

- If $N_\mathbb{R}$ is spanned by the $u_\rho$ for $\rho \in \Sigma(1)$, then

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \text{Cl}(X_\Sigma) \longrightarrow 0$$

is exact.

- Hit this with $\text{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$ get

$$0 \longrightarrow G \longrightarrow (\mathbb{C}^*)^{\Sigma(1)} \longrightarrow T_N$$

- Since $G \subseteq (\mathbb{C}^*)^{\Sigma(1)}$, $G$ acts on $\mathbb{C}^{\Sigma(1)}$

- Modulo some (hard) technical details about quotients by algebraic group actions (GIT) we get $X_\Sigma$ as the quotient modulo this action
Sanity check, II

For $\mathbb{P}^2$, the first exact sequence above is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix} : \mathbb{Z}^2 \longrightarrow \mathrm{Cl}(\mathbb{P}^2) \longrightarrow 0$$

After taking $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$, we have

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} : (\mathbb{C}^*)^2 \longrightarrow (\mathbb{C}^*)^3 \longrightarrow \mathbb{C}^* \longrightarrow 0$$

And indeed $\mathbb{C}^*$ acts on $\mathbb{C}^3$ via $a \sim \lambda a$. 
Why is this important?

**Theorem**

Any coherent sheaf on a toric variety $X_\Sigma$ comes from a finitely-generated $\text{Cl}(X_\Sigma)$-graded module over the “Cox ring”

$$S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$$

(Generalizes a result of Serre in the case of $X_\Sigma = \mathbb{P}^n$, where the “Cox ring” is the same as the homogeneous coordinate ring $\mathbb{C}[x_0, \ldots, x_n]$.)

Translates abstract questions about sheaf cohomology into *concrete, computable* questions in multigraded commutative algebra(!)