Affine and Projective Toric Varieties
Math in the Mountains Tutorial

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Outline

The torus, characters

- $T = (\mathbb{C}^*)^n$ is an abelian group under coordinatewise multiplication, *and* an algebraic variety. Ex. $\mathbb{C}^* \leftrightarrow V(xy-1) \subset \mathbb{C}^2$
- A *character* of T is morphism of varieties and group homomorphism $\chi : T \to \mathbb{C}^*$
- Any character of T has the form

$$(t_1, \ldots, t_n) \mapsto \prod_i t_i^{a_i}$$

  for some $a_i \in \mathbb{Z}$.
- Write $M = \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{Z}\} \simeq \mathbb{Z}^n$, the *character lattice* of T.
- *Mnemonic: M for "monomial map"*

One-parameter subgroups

- A one-parameter subgroup of T is a morphism of varieties and group homomorphism $\lambda : \mathbb{C}^* \to T$
- Any such mapping has the form

$$t \mapsto (t^{a_1}, \ldots, t^{a_n}) \text{ for some } a_i \in \mathbb{Z}.$$

- So we have a second lattice $N \simeq \mathbb{Z}^n$, the lattice of 1-parameter subgroups, and $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ is the associated torus
- Given $m \in M$ and $n \in N$ we get $\chi^m : T \to \mathbb{C}^*$ and $\lambda^n : \mathbb{C}^* \to T$
- Gives a pairing $\langle\ ,\ \rangle : M \times N \to Z$ via

$$\chi^m \circ \lambda^n : \mathbb{C}^* \to \mathbb{C}^*$$
$$t \mapsto t^{\langle m,n \rangle}$$

A theorem and a definition

Theorem (Sumihiro)

1. *If T_1 and T_2 are tori and $\varphi : T_1 \to T_2$ is a morphism of varieties and group homomorphism, then $\mathrm{im}(\varphi)$ is a closed torus in T_2*
2. *If H is an irreducible subvariety of a torus T that is a subgroup of T, then H is a torus.*

Definition

*An **affine toric variety** is an irreducible variety containing T as a Zariski open subset, and such that the action of T on itself extends to a morphism $T \times V \to V$.*

First Examples

- $T = (\mathbb{C}^*)^n$ itself
- $\mathbb{C}^n$, $\mathbb{P}^n$, via the "obvious" inclusions $T \subset \mathbb{C}^n \subset \mathbb{P}^n$
- The cuspidal cubic $V(y^2 - x^3) \subset \mathbb{C}^2$. The torus T in this case is $T = \{(t^2, t^3) \mid t \in \mathbb{C}^*\}$

“Why do we care?”

The action of T lets us break things into simple bits –

- The “three-fold path:"
  - Lattice points (free abelian groups of finite rank)
  - Toric ideals
  - Affine semigroups
- If T_N has character lattice M and $\mathcal{A} = \{a_1, \ldots, a_n\} \subset M$, let

$$\begin{aligned} \varphi_{\mathcal{A}} : T &\to \mathbb{C}^n \\ t &\mapsto (\chi^{a_1}(t), \ldots, \chi^{a_n}(t)) \end{aligned}$$

Lattice points and the variety $Y_{\mathcal{A}}$

- Here is the formal definition of the variety defined by a monomial parametrization:

Definition

$Y_{\mathcal{A}} = \overline{\varphi_{\mathcal{A}}(T)}$, where the bar is Zariski closure.

Proposition

$Y_{\mathcal{A}}$ is an affine toric variety with character lattice $\mathbb{Z}(\mathcal{A})$. The dimension of $Y_{\mathcal{A}}$ is the rank of the lattice $\mathbb{Z}(\mathcal{A})$.

- Example: $M = \mathbb{Z}$, $\mathcal{A} = \{2, 3\}$ gives $Y_{\mathcal{A}} = V(y^2 - x^3)$

The ideal of $Y_{\mathcal{A}}$

- For $\mathcal{A}$ as above, consider $\mathbb{Z}^n \to M$ defined by $e_i \mapsto a_i$.
- Let L be the kernel of this map, so $0 \to L \to \mathbb{Z}^n \to M$ is exact
- Write each $\ell \in L$ as $\ell = \ell_+ - \ell_-$ (that is, $\ell_+ = \sum_{\ell_i > 0} \ell_i e_i$ and $\ell_- = -\sum_{\ell_i < 0} \ell_i e_i$)
- Then $x^{\ell_+} - x^{\ell_-}$ vanishes on the image of $\varphi_{\mathcal{A}}$, hence is in $I(Y_{\mathcal{A}})$.
- In fact we have,

Theorem

$$I(Y_{\mathcal{A}}) = \langle x^{\alpha} - x^{\beta} \mid \alpha, \beta \in \mathbb{N}^n, \alpha - \beta \in L \rangle.$$

Toric ideals

- As we saw in the statistics talk, ideals of this form have a name:

Definition

*A **toric ideal** is a prime lattice ideal*

$$\langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^n, \alpha - \beta \in L\rangle$$

for $L \subset \mathbb{Z}^n$.

- In fact, the binomial form of the generators is enough:

Theorem

$I \subset \mathbb{C}[x_1, \ldots, x_n]$ is toric $\Leftrightarrow$ I is prime and generated by binomials.

Two Examples

- Consider $\mathbb{Z}^2 \to \mathbb{Z}$ defined by columns of $\mathcal{A} = (2 \;\; 3)$. (Note: we'll sometimes be sloppy about what $\mathcal{A}$ represents – it could be the set $\{2, 3\}$ or the 1×2 integer matrix with those as the columns(!))
- The kernel is $L = \langle(3 \;\; -2)\rangle$
- Gives binomial $x^3 - y^2$, which generates the toric ideal
- Similarly if
$$\mathcal{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$
then $L = \langle(1 \;\; -2 \; 1)\rangle$ and we get $I = \langle xz - y^2\rangle$, so $V(I)$ is a quadric cone.

Affine semigroups

- A semigroup is a set with a binary operation satisfying all of the group axioms except for the existence of inverses.
- Affine semigroups are abelian, finitely generated, contained in a lattice, operation written as $+$
- If S is a semigroup, we can define a *semigroup algebra*

$$\mathbb{C}[S] = \mathrm{Span}\{x^\alpha : \alpha \in S\} \text{ with } x^\alpha \cdot x^\beta = x^{\alpha+\beta}$$

- It follows that

Theorem

$\mathbb{C}[S]$ *is an integral domain, finitely generated as a* $\mathbb{C}$*-algebra, hence corresponds to an affine variety* $\mathrm{Spec}\ \mathbb{C}[S]$*. This variety is toric; if* $S = \mathbb{N}\mathcal{A}$ *for* $A \subset M$*, then it is isomorphic to* $Y_{\mathcal{A}}$*.*

An example

Let S be the affine semigroup in $\mathbb{Z}^2$ generated by $(2,0),(1,1),(0,2)$.

The semigroup algebra $\mathbb{C}[S] = \mathbb{C}[s^2, st, t^2]$

The corresponding variety $\mathrm{Spec}\ \mathbb{C}[S]$ is the affine toric variety $Y_{\mathcal{A}}$ for $\mathcal{A} = \{(2,0),(1,1),(0,2)\}$.

This is the quadric cone $V(xz - y^2)$ from before.

Three ways to describe affine toric varieties

From the last example, the next statement should be plausible at least:

Theorem

The following are equivalent:

- V *is an affine toric variety*
- $V = Y_{\mathcal{A}}$ *for some* $\mathcal{A} \subset M$
- $V = V(I)$ *for a toric ideal*
- $V = \mathrm{Spec}\ \mathbb{C}[S]$ *for an affine semigroup* S.

Cones

- Let $F \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ be a finite set. We define the *cone spanned by F* to be:

$$\operatorname{cone}(F) = \left\{ \sum_{u \in F} \lambda_u u \mid \lambda_u \geq 0 \right\}$$

- The *convex hull* of S is

$$\operatorname{conv}(F) = \left\{ \sum_{u \in F} \lambda_u u \mid \lambda_u \geq 0, \sum_{u \in F} \lambda_u = 1 \right\}$$

Example

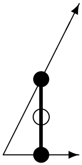


Figure: $F = \{(1,0),(1,2)\}$ in $\mathbb{Z}^2$

$\mathrm{cone}(F)$ is the region between the two rays; $\mathrm{conv}(F)$ is the thicker vertical line segment.

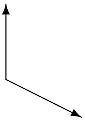
The dual cone

Given a cone σ in $N_{\mathbb{R}}$, the dual cone is

$$\sigma^{\vee} = \{n \in M_{\mathbb{R}} \mid \langle m, n \rangle \geq 0 \forall n \in \sigma\}$$

For $\sigma = \mathrm{cone}(\{(1,0),(1,2)\})$ as on the previous slide, the dual cone is

$$\sigma^{\vee} = \mathrm{cone}(\{(0,1),(2,-1)\})$$



Some terminology

- A cone $\sigma = \mathrm{cone}(F)$ is *rational* if $F \subset N$; *polyhedral* if F is finite (no "ice cream cones!"); *strongly convex* if the largest linear subspace contained in σ is $\{0\}$
- RPC = rational polyhedral cone, SCRPC = strongly convex, rational, polyhedral cone
- A SCRPC is *smooth* if the minimal generators of the rays of the cone are a subset of a $\mathbb{Z}$-basis of N
- A SCRPC is *simplicial* if the minimal generators are a subset of a basis of $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$

An important fact

Theorem (Gordan's Lemma)

If σ is a RPC in N, then $S_\sigma = \sigma^\vee \cap M$ is a finitely-generated semigroup.

Proof: $\sigma^\vee$ is also an RPC, so $\sigma^\vee = \mathrm{cone}(T)$ for T a finite subset of M. The parallelotope $K = \{\sum_{m \in T} \lambda_m m \mid 0 \leq \lambda_m < 1\}$ is bounded, so $|K \cap M|$ is finite and $T \cup (K \cap M) \subset S_\sigma$. The key point is $T \cup (K \cap M)$ generates S_σ since if $w \in S_\sigma$,

$$w = \sum_m \lambda_m m = \sum \lfloor \lambda_m \rfloor m + \sum (\lambda_m - \lfloor \lambda_m \rfloor) m$$

The first term is in T, the second is in K. //

A corollary, saturated semigroups

- Let $S_\sigma = \sigma^\vee \cap M$; by the previous result:

Corollary

Let $\sigma \subset N_{\mathbb{R}}$ be an RPC. Then $\operatorname{Spec} \mathbb{C}[S_\sigma]$ is an affine toric variety, denoted U_σ. It has dimension $n = \operatorname{rank}(N)$ if and only if σ is a SCRPC.

- An affine semigroup $S \subset M$ is said to be *saturated* if $km \in S \Rightarrow m \in S$ for $k \in \mathbb{N}$.
- Example: $S = \langle (4,0),(3,1),(1,3),(0,4) \rangle$ is *not saturated* (why not?)

Normality, smoothness, etc.

Theorem

Let V be an affine toric variety with torus T_N. The following are equivalent:

- *V is normal (coordinate ring is integrally closed)*
- *$V = \mathrm{Spec}\ \mathbb{C}[S]$ for a saturated affine semigroup S*
- *$V = \mathrm{Spec}\ \mathbb{C}[S_\sigma]$ for a SCRPC $\sigma \subset N_{\mathbb{R}}$*

Theorem

U_σ is a smooth variety if and only if σ is a smooth cone.

Points, morphisms

A point $p \in \operatorname{Spec} \mathbb{C}[S] \leftrightarrow$ a semigroup homomorphism γ in $\operatorname{Hom}_{sg}(S, \mathbb{C})$ via

$$m \in S \mapsto \chi^m(p) \in \mathbb{C}.$$

Intrinsically, if $p = m \mapsto \gamma(m)$, then for all $t \in T$, $t \cdot p : m \mapsto \chi^m(t)\gamma(m)$.

Definition

*A morphism $V_1 \to V_2$ is **toric** if $\mathbb{C}[S_2] \to \mathbb{C}[S_1]$ is induced by a semigroup homomorphism $S_2 \to S_1$.*

Theorem

$\varphi : V_1 \to V_2$ is toric if and only if $\varphi(T_{N_1}) \subseteq T_{N_2}$ and φ restricts to a group homomorphism on T_{N_1}. Toric morphisms are T-equivariant: $\varphi(t \cdot p) = \varphi(t)\varphi(p)$.

The basics

- $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$ where $(x_0, \ldots, x_n) \sim \lambda \cdot (x_0, \ldots, x_n)$ for all $\lambda \in \mathbb{C}^*$
- The torus in $\mathbb{P}^n$ is the complement of $V(x_0 \cdots x_n)$
- Projective parametric toric varieties:

Definition

If $\mathcal{A} = \{m_1, \ldots, m_{n+1}\} \subset M$, we have

$$T_N \xrightarrow{\varphi_{\mathcal{A}}} (\mathbb{C}^*)^{n+1} \longrightarrow \mathbb{C}^{n+1} \longrightarrow \mathbb{P}^n$$

The image is the projective toric variety $X_{\mathcal{A}} = \overline{\varphi_{\mathcal{A}}(T_N)}$.

Two subtle points

- Since $1 \to \mathbb{C}^* \to (\mathbb{C}^*)^{n+1} \to T_N \to 1$ is exact, the character lattice $M_{\mathbb{P}^n}$ is $\{(a_0, \ldots, a_n) \mid \sum_i a_i = 0\} = \mathbb{Z}^{n+1}/\mathbb{Z}(1, \ldots, 1)$
- Two rather different parametrizations can give the same projective toric variety:
- Example A: $M = \{0, 1, \ldots, n\}$ in $\mathbb{Z}$ gives $\mathbb{C}^* \to (\mathbb{C}^*)^{n+1} \to \mathbb{P}^n$ defined by $t \mapsto (1, t, \ldots, t^n)$
- Example B: $M = \{(n, 0), (n-1, 1), \ldots, (0, n)\} \subset \mathbb{Z}^2$ gives $(s, t) \mapsto (s^n, s^{n-1}t, \ldots, t^n)$
- The projective varieties in A and B are the same, but the affine varieties are not (parametrization of $Y_{\mathcal{A}}$ is not homogeneous; for $Y_{\mathcal{B}}$, it is)

Connection with affine torics

Definition

Given $\mathcal{A} \subset M$, let $\mathbb{Z}^1(\mathcal{A}) = \{\sum_i a_i m_i \mid \sum_i a_i = 0\}$

Theorem

The following are equivalent

- *$Y_{\mathcal{A}}$ is the affine cone over the projective variety $X_{\mathcal{A}}$*
- *$I(Y_{\mathcal{A}})$ is a homogeneous ideal*
- *There exists a $u \in N$ and $k \in \mathbb{N}$ such that $\langle m_i, u \rangle = k$ for all i. (Note: this says the m_i all lie on a hyperplane in M.*

Consequences

Theorem

The character lattice of $X_{\mathcal{A}}$ is $\mathbb{Z}^1(\mathcal{A})$. This has rank $\mathrm{rank}\ \mathbb{Z}(\mathcal{A}) - 1$ if the m_i all lie on a hyperplane and $\mathrm{rank}\ \mathbb{Z}(\mathcal{A})$ otherwise.

Example: If $\mathcal{A}$ consists of the columns of

$$\begin{pmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{pmatrix},$$

then $\mathrm{rank}\mathbb{Z}^1(\mathcal{A}) = 1$ and $X_{\mathcal{A}}$ is a curve.

Affine charts

- In $\mathbb{P}^n$, let $U_i = \{x_i \neq 0\}$ – a covering of $\mathbb{P}^n$ by affines isomorphic to $\mathbb{C}^n$.
- If $X \subset \mathbb{P}^n$ is a variety, let $X_i = X \cap U_i$.
- In toric case, since $x_i = \chi^{m_i}$, passing to $U_i = \operatorname{Spec} \mathbb{C}[x_0/x_i, \ldots, x_n/x_i]$ is equivalent to considering $A_i = \{m_j - m_i \mid j \neq i\}$
- Have

Theorem

$X_{\mathcal{A}} \cap U_i = Y_{\mathcal{A}_i} = \operatorname{Spec} \mathbb{C}[S_i]$, where $S_i = \mathbb{N}\mathcal{A}_i$.

- In fact, can do better:

$$X_{\mathcal{A}} = \bigcup_{m_i \text{ a vertex of } \operatorname{conv}(\mathcal{A})} X_{\mathcal{A}} \cap U_i$$

An example, projective normality

- For the rational quartic from $\mathcal{A} = \{(4,0),(3,1),(1,3),(0,4)\}$, we have

$$X_{\mathcal{A}} \cap U_0 = \mathrm{Spec}\ \mathbb{C}[1, t/s, t^3/s^3, t^4/s^4] = \mathbb{C}[t/s]$$

- So $X_{\mathcal{A}} \cap U_0$ is $\mathbb{C}$.
- Similarly for $U_3 \Rightarrow$ a smooth, projective, rational curve.
- Applying a standard notion from algebraic geometry:

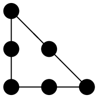
Definition

*$X_{\mathcal{A}}$ is said to be **projectively normal** if the affine cone $Y_{\mathcal{A}}$ is normal (for toric, $\Leftrightarrow$ saturated)*

- Exercise: the rational quartic above is not projectively normal

Polytopes

- A polytope P can be defined either as the convex hull of a finite set of points, *or* as a compact intersection of finitely many half-spaces
- The lattice polytope



is

$$\operatorname{conv}(\{(0,0),(2,0),(0,2)\}$$

and

$$\{x \geq 0\} \cap \{y \geq 0\} \cap \{x+y \leq 2\}.$$

Polytope terminology

Definition

Let P be a lattice polytope in M

- *A hyperplane $H = \{m \mid \langle m, n_F \rangle = -a_F\}$ is a* ***supporting hyperplane*** *if $P \subset \{m \mid \langle m, n_F \rangle \geq -a_F\}$*
- *$P \cap H$ for a supporting hyperplane is a* ***face*** *of P; dimension of a face is dimension of the affine span*
- *P is a* ***simplex*** *if P has $\dim(P) + 1$ vertices*
- *P is* ***simplicial*** *if each facet (codim 1 face) is a simplex*
- *P is* ***simple*** *if each vertex lies in $\dim(P)$ facets.*

Two examples

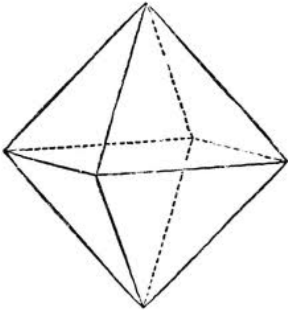


Figure: simplicial $6, 8, 12$

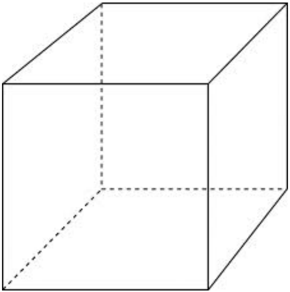


Figure: simple $8, 12, 6$

Normal polytopes

- The definition is:

Definition

*A lattice polytope $P \subset M_{\mathbb{R}}$ is **normal** if*

$$(kP \cap M) + (\ell P \cap M) = (k+\ell)P \cap M$$

for all $k, \ell \geq 1$. (Note: The $\subseteq$ here is "automatic.")

- Exercise: $P = \mathrm{conv}(\{0, e_1, e_2, e_1 + e_2 + 2e_3\})$ is not normal.
- A nice fact:

Theorem

If P is a lattice polytope with $\dim(P) = n \geq 2$, then kP is normal for all $k \geq n-1$.

The toric variety of a polytope

The key idea here is that charts $\Leftrightarrow$ vertices of P. Given a lattice polytope $P \subset M$ we get a cone and a semigroup from each vertex m_i of P as follows:

Definition

- *The vertex cone is*

$$\sigma_i = \mathrm{cone}(P \cap M - m_i)^\vee \subseteq N$$

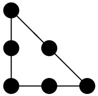
(Note: The notation $P \cap M - m_i$ means: translate $P \cap M$ to place m_i at the origin!)

- *The semigroup is*

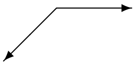
$$S_{P,v} = \mathbb{N}(P \cap M - v) \subseteq M$$

An example will make all clear, we hope ...

Let P be the following polytope from before:

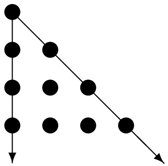


and take $v = 2e_2 = (0, 2)$. The cone σ_i is $\mathrm{cone}(\{e_1, -e_1 - e_2\})$:



Example, continued

The lattice points in M making up the semigroup $S_{P,v}$ are shown below: top vertex is translated to the origin:



The affine charts

Theorem

$X_{P\cap M} \cap U_i = \operatorname{Spec} \mathbb{C}[\sigma_i^\vee \cap M]$

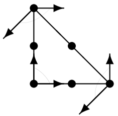
Definition

*P is **very ample** if for all vertices v, $S_{P,v}$ is saturated.*

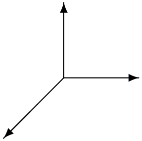
Exercise: Let $P = \operatorname{conv}\{0, e_1, e_2, e_1 + e_2 + 2e_3\}$ from before. Show $|P \cap M| = 4$ so $X_{P\cap M} \to \mathbb{P}^3$. Show that $X_{P\cap M}$ is not smooth. Show that P is not very ample.

How it all fits together, ...

Let's draw in the vertex cones at each vertex of the lattice polytope in the previous example:



If we "reassemble" these cones with vertices at the origin, we see they fit together to cover $N_{\mathbb{R}} = \mathbb{R}^2$. Will return to this in the next talk(!)



The toric variety of a polytope P

Note that we have only defined $X_{P\cap M}$ so far.

Definition

Let P be a lattice polytope in M. The toric variety of P $X_P = X_{kP\cap M}$ where kP is very ample.

Corollary

For an n-dimensional polytope, $(n-1)P$ always gives an embedding of X_P in $\mathbb{P}(V)$.

Final comments

- Warning: There are smooth, complete varieties that are not projective (Shafarevich example for surfaces). Such examples exist for toric varieties, too, but only in dimensions ≥ 3.
- Motivated by the last example,

Definition

*The **normal fan** of a polytope P is the collection of cones glued like a simplicial complex, generated by the inward normals to the faces.*

- The point is that P and kP have the same normal fan, so they give, intrinsically the same (abstract) toric variety, to be described in the next talk.