Software for Computations with Toric Varieties
Math in the Mountains Tutorial

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Outline

1. Cones, polytopes, lattice points
2. Toric varieties and associated structures

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- Small enough, in fact, that we have been able to do almost everything *by hand or by eye*
- This will not be true for larger examples, of course!
- But fortunately, there are a number of powerful open-source software packages that can do most of the calculations one might want, and we want to say a few words about them here.

`4ti2`

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- Unix/linux, MacOS, and Windows versions available
- Basic functions include: Markov bases (i.e. binomial basis for toric ideal defined by a given matrix $\mathcal{A}$), Gröbner and Graver bases for toric ideals, Hilbert bases for cones, etc.

Some examples

3×3 integer magic squares (equal rows sums, column sums, and diagonals) correspond to the integer solutions of

$$\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix} u = \mathcal{A}u = 0$$

Magic square example, cont.

To read this into `4ti2`, we would create a text file `3x3magic.mat` with contents

```
7 9
1 1 1 -1 -1 -1  0  0  0
1 1 1  0  0  0 -1 -1 -1
0 1 1 -1  0  0 -1  0  0
1 0 1  0 -1  0  0 -1  0
1 1 0  0  0 -1  0  0 -1
0 1 1  0 -1  0  0  0 -1
1 1 0  0 -1  0 -1  0  0
```

Markov basis for the toric ideal $I_{\mathcal{A}}$

Running `4ti2` to compute the Markov basis produces a new text output file called something like `3x3magic.mar`

```
9 9
 0  0  0  0  1 -1  0 -1  1
 0  0  0  1 -1  0 -1  1  0
 0  0  0  1  0 -1 -1  0  1
 0  1 -1  0 -1  1  0  0  0
 0  1 -1  0  0  0  0 -1  1
 1 -1  0 -1  1  0  0  0  0
 1 -1  0  0  0  0 -1  1  0
 1  0 -1 -1  0  1  0  0  0
 1  0 -1  0  0  0 -1  0  1
```

What this means

- Note how each row here gives a matrix like (row 1):

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- As a “move” that would take one magic square to another magic square with the same row and column sum(!)

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- For instance say $\sigma = \mathrm{cone}(e_1, e_1 + 4e_2)$.
- To put this into a form `4ti2` can work with, we essentially need to think of the dual cone, or write σ as the intersection of the half-spaces defined by $\langle e_2, (x, y)\rangle = y \geq 0$ and $\langle 4e_1 - e_2, (x, y)\rangle = 4x - y \geq 0$.

Example, continued

For this computation, we actually need to prepare several input text files, one for the matrix of coefficients on the left sides of the inequalities `cone.mat`

```
2 2
0 1
4 -1
```

One for the inequalities `cone.rel`:

```
1 2
> >
```

and one for the right hand sides `cone.rhs`

```
1 2
0 0
```

Hilbert basis

Running `zsolve` or `hilbert` will yield an output file `cone.zhom` (or `cone.hil`) containing what we are looking for:

```
5 2
1  0
1  4
1  1
1  2
1  3
```

Other packages – a (very) incomplete list

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- `LattE`, `LattE integrale`, `LattE for tea too`: developed by a team at UC, Davis led by Jesus deLoera, with M. Köppe; `www.math.ucdavis.edu/ latte/` has some overlap with `4ti2`, and now a combined version. Additional functionality for enumeration of lattice points in polytopes, Ehrhart polynomials, integrals of polynomial functions over polytopes, volume computations, ...

The incomplete list, continued

- `Normaliz` by W. Bruns, B. Ichim and C. Söger, `www.mathematik.uni-osnabrueck.de/normaliz/` – dual cones, triangulations, Hilbert bases, lattice points of a lattice polytope, the normalization of an affine monoid Hilbert (or Ehrhart) series.

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- `polymake` by Gawrilov and Joswig, `http://www.polymake.org/doku.php`: "a tool to study the combinatorics and the geometry of convex polytopes and polyhedra. It is also capable of dealing with simplicial complexes, matroids, polyhedral fans, graphs, tropical objects, and other objects"

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- See Appendix B of [CLS] for several extensive examples

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