Toric Varieties in Geometric Modeling
Math in the Mountains Tutorial

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Outline

1. Geometric modeling – basic examples
2. Toric surface patches
3. The (algebraic) moment map
4. Linear precision and connections with statistics

Computer aided geometric design

- Since the 1960's Bézier curves and surfaces have been a fundamental tool for designing, rendering, manufacturing shapes
- Work by Bézier (Renault) and de Casteljau (Citroën)
- We will see how the study of toric varieties gives some interesting tools for understanding this area and developing new primitives, and
- discuss some (perhaps unexpected) connections with moment maps and Birch's Theorem from the talk on algebraic statistics

Example 0 – cubic Bézier curves

- Let P_0, P_1, P_2, P_3 be any four points in $\mathbb{R}^2$ (same construction would also work in $\mathbb{R}^3$, or $\mathbb{R}^d$ more generally)
- The associated Bézier cubic is the parametric curve defined by

$$b(t) = (x(t), y(t)) = \sum_{i=0}^{3} \binom{3}{i} t^i (1-t)^{3-i} P_i$$

(Exercise: This is always a subset of a rational algebraic curve of degree 3 in x, y.)

- Note: the coefficients are the Bernstein basis for polynomials of degree ≤ 3 and

$$\sum_{i=0}^{3} \binom{3}{i} t^i (1-t)^{3-i} = (t + (1-t))^3 = 1$$

Two Bézier curves

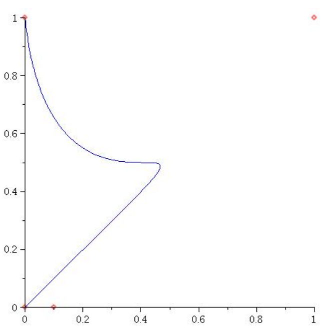

Figure: $(0,0),(1,1),(0.1,0),(0,1)$

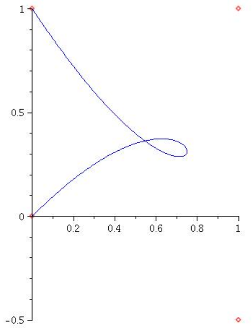

Figure: $(0,0),(1,1),(1,-0.5),(0,1)$

Bézier curves, cont.

- Note the way the P_i (the *control points*) *control the shape*
- $b(0) = P_0$ and $b(1) = P_3$
- the relation on the Bernstein polynomials given above implies that $b(t)$ lies in $\mathrm{conv}\{P_0, P_1, P_2, P_3\}$ for all $0 \le t \le 1$
- Tangent vector at P_0 is determined by $P_1 - P_0$; tangent vector at P_3 is determined by $P_3 - P_2$
- More complicated shapes (e.g. character outlines in a typeface) can be specified, or even designed, via Bézier *splines*.
- Can also replace cubic Bézier curves by degree n Bézier curves with $n + 1$ control points (splines are generally superior for applications, though!)

Bézier surfaces

- Can consider similar parametric surfaces
- Two commonly-used types: Bézier triangles and rectangles (“tensor products” of two Bézier curves)
- For example, a Bézier triangle of order 3 would be given by the selection of 10 control points P_{ij} in $\mathbb{R}^3$:

$$b(x,y) = \sum_{i=0}^{3} \sum_{j=0}^{3-i} \frac{3!}{i!j!(3-i-j)!} x^i y^j (1-x-y)^{3-i-j} P_{ij}$$

  on the domain $\Delta = \mathrm{conv}\{(0,0),(1,0),(0,1)\}$
- Note: the edges will be cubic Bézier curves defined by the 4-tuples of control points $\{P_{00}, P_{10}, P_{20}, P_{30}\}$, $\{P_{00}, P_{10}, P_{20}, P_{30}\}$, and $\{P_{30}, P_{21}, P_{12}, P_{03}\}$

A Bézier triangle

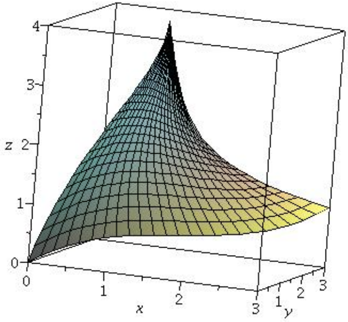

Figure: Corner control points are $(0,0,0), (3,3,1), (1,2,4)$

A generalization

- In [K], Krasauskas introduced a generalization of the Bézier triangles and tensor product patches:
- Construction starts from any convex lattice polygon Δ in $\mathbb{R}^2$
- Number the edges in some way with $i = 1, \ldots, r$; say $\mathbf{v}_i$ is an inward normal to the ith edge, and $h_i(x, y) = \langle \mathbf{v}_i, (x, y)\rangle + a_i = 0$ defines that line
- The article [K] defines *toric Bézier basis, or blending, functions* indexed by the lattice points $m \in \Delta \cap \mathbb{Z}^2$:

$$F_m(x, y) = h_1(x, y)^{h_1(m)} \cdots h_r(x, y)^{h_r(m)}$$

Toric patch parametrizations

Let $w_m > 0$ be weights, and choose control points $\mathcal{P} = \{P_m \in \mathbb{R}^3 \mid m \in \mathcal{A}\}$, both indexed by $\mathcal{A} = \Delta \cap \mathbb{Z}^2$. The corresponding toric surface patch of shape $\mathcal{A}, w$ is the rational parametric surface

$$b_{\mathcal{A},w,\mathcal{P}}(x,y) = \frac{1}{\sum_{m\in\mathcal{A}} w_m F_m(x,y)} \sum_{m\in\mathcal{A}} w_m F_m(x,y) P_m$$

for $(x,y) \in \Delta$. (We may sometimes omit the $\mathcal{P}$ in the notation.)

Example – tensor product surfaces as toric patches

- Use the normal vectors $(0,1),(-1,0),(0,-1),(1,0)$ counterclockwise around Δ
- Corresponding h_i are $h_1(x,y) = y$, $h_2(x,y) = p - x$, $h_3(x,y) = q - y$, $h_4(x,y) = x$
- We take $w_m = \binom{p}{a}\binom{q}{b}$ and

$$F_m(x,y) = x^a(p-x)^{p-a} \cdot y^b(q-y)^{q-b}$$

- A reparametrization of the usual tensor product Bézier patch of bidegree (p,q) – standard form scales by letting $x = p\xi$ and $y = q\eta$ for $(\xi,\eta) \in [0,1] \times [0,1]$.

Example – Bézier triangles as toric patches

- Let $\Delta = \mathrm{conv}\{(0,0),(k,0),(0,k)\}$ and
- Let $\mathcal{A} = \Delta \cap \mathbb{Z}^2$ be the set of all lattice points in Δ
- Exercise: $b_{\mathcal{A},w}$ for $\Delta_k = \mathrm{conv}\{(0,0),(k,0),(0,k)\}$ is (a reparametrization of) the Bézier triangle from before (with proper choice of w_m).

General properties

- Exercises: Just as in the simpler cases,
- Image $b_{\mathcal{A},w,\mathcal{P}}(\Delta)$ lies in the convex hull of the set $\mathcal{P}$,
- Image *contains* the control points corresponding to vertices of Δ
- Control points for lattice points of Δ on edges, but not at vertices, determine shape of the boundaries; control points for interior lattice points of Δ can be used to introduce concavity, ...
- “Structural” singular points of the image (i.e. present for generic control points) are determined by lattice geometry of Δ (singular cones in normal fan)
- For more about all of this, see [CGS].

But does the generality get you anything?

- In the applications, triangles and rectangles can be slightly awkward: for some surfaces, might need to subdivide *a lot*
- Being able to construct 5- or 6-sided patches, for example, might be useful for some things
- Consider a toric surface patch from the hexagon

$$\Delta = \mathrm{conv}\{(0,0),(1,0),(2,1),(2,2),(1,2),(0,1)\}$$

for some particular control points in $\mathbb{R}^3$. The interior lattice point $(1,1)$ corresponds to a control point at $(-3,-7,5)$

The F_m for this $\mathcal{A}$

$$
\begin{aligned}
F_{(0,0)} &= (y-x+1)(2-x)^2(2-y)^2(x-y+1)\\
F_{(1,0)} &= (2-x)(2-y)^2(x-y+1)^2x\\
F_{(2,1)} &= x^2y(2-y)(x-y+1)^2\\
F_{(2,2)} &= (y-x+1)x^2y^2(x-y+1)\\
F_{(1,2)} &= xy^2(y-x+1)^2(2-x)\\
F_{(0,1)} &= y(y-x+1)^2(2-x)^2(2-y)\\
F_{(0,0)} &= y(y-x+1)(2-x)(2-y)(x-y+1)x.
\end{aligned}
$$

Also, let $w_m = 1$ for all m

Two views of a toric surface patch

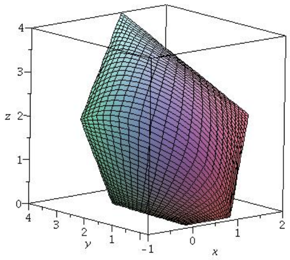

Figure: Hexagonal toric patch

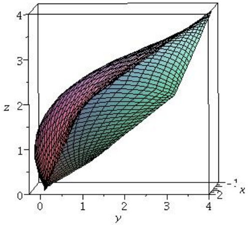

Figure: Rotated view showing concavity

Partial patches

We could also do the same replacing $\mathcal{A}$ by *any subset* $\mathcal{A}'$ of the lattice points in Δ (usually want to include all of the vertices so the polygon itself does not change)

Exercise: What is the relation between the corresponding toric surface patches?

A message from our sponsor

- From what we have seen previously and these examples, it should be relatively clear that the image of $b_{\mathcal{A},w,\mathcal{P}}$ is somehow related to a *toric variety*. But, what is the precise relation?
- To simplify notation, let $\ell = |\mathcal{A}|$, take $w_m = 1$ all m
- First observation: Since h_m come from the inward normals to Δ, the map $H : \Delta \to \mathbb{R}^\ell$ given by the h_m has image in $\mathbb{R}^\ell_{\geq 0}$.
- The toric blending functions come from composing this H with $\chi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ defined by

$$y \longmapsto (y_1^{h_1(m)} \cdots y_\ell^{h_\ell(m)} : m \in \mathcal{A})$$

The "punch line"

- So far, we have $\chi \circ H : \Delta \to \mathbb{R}^{\ell}_{\geq 0} \to \mathbb{R}^{\ell}_{\geq 0}$. The toric surface patch is the composition $\pi_P \circ \chi \circ H$, where $\pi_P : \mathbb{R}^{\ell} \to \mathbb{R}^3$ is the affine form of a linear projection defined by the set of control points
- (Exercise) Since $h_i(x, y) = \langle \mathbf{v}_i, (x, y)\rangle + a_i$, the m-component of $\chi(y)$ is just $y^a z^m$, where $a = (a_1, \ldots, a_\ell)$ comes from the constant terms, and $z = (z_j)$ where

$$z_j = \prod_{i=1}^{\ell} y_i^{\langle \mathbf{v}_i, \mathbf{e}_j \rangle}, \quad j = 1, 2$$

- But the map $z \mapsto (z^m : m \in \mathcal{A})$ is just the monomial parametrization of the toric variety $X_{\mathcal{A}}$.

Factoring toric patches another way

- Hence a toric surface patch can be factored a different way as

$$\Delta \to (X_{\mathcal{A}})_{\geq 0} \to \mathbb{R}^3$$

- The first map is $\chi \circ H$
- The second is the projection π_P defined by the control points
- also, $(X_{\mathcal{A}})_{\geq 0}$ is the is the set of points obtained from the monomial parametrization of $X_{\mathcal{A}}$ by taking all parameter values real and ≥ 0.

Relation with earlier examples

- If Δ_k is $\mathrm{conv}\{(0,0),(k,0),(0,k)\}$ as above, then $(X_{\Delta_k})_{\geq 0}$ is a subset of the real points of the degree-k *Veronese image* of $\mathbb{P}^2$
- $(X_{[p,0]\times[0,q]})_{\geq 0}$ is a subset of the Veronese-Segre bidegree p, q image of $\mathbb{P}^1 \times \mathbb{P}^1$
- The images of corresponding toric surfaces patches will be projections of these defined by the control points
- Krasauskas' construction generalizes this to any Δ, though, so it's very flexible!

But wait a minute, ...

- Will it *always* be true that a toric surface patch preserves the shape of Δ to the degree we saw with the hexagon?
- For instance, do k-sided polygons Δ map to k-sided $b_{\mathcal{A},w}(\Delta)$? Of course, it *also* depends on the choice of the control points, but there is an interesting connection between this applied question and a general theoretical statement about toric varieties (which in turn connects with interesting questions in topology and symplectic geometry)
- The connection depends on an algebraic version of the *moment map*

The moment map

Let $\mathcal{A}$ be a set of lattice points with $\Delta = \mathrm{conv}(\mathcal{A})$. The (algebraic) moment map of the toric variety $X_\mathcal{A}$ is the mapping $X_\mathcal{A}$ defined as follows

$$f : X_\mathcal{A} \longrightarrow \mathbb{R}^2$$
$$x \longmapsto \frac{1}{\sum_{m\in\mathcal{A}} |x_m|} \sum_{m\in\mathcal{A}} |x_m| m$$

Recall the entries of points in $X_\mathcal{A}$ are in 1-1 correspondence with the $m \in \mathcal{A}$ via the monomial parametrization. Also, the usual symplectic moment map is similar but with $|x_m|^2$ instead of $|x_m|$.

The theorem

Theorem

The mapping f restricts to a homeomorphism between the non-negative real part of $X_{\mathcal{A}}$ and the polygon Δ.

(See Theorem 12.2.2 in [CLS] or Chapter 4 of [F].)

As we said before, for any collection of control points, the image of the toric surface patch is a linear projection of the non-negative real part of $X_{\mathcal{A}}$.

Final observations

- Looking back at the algebraic moment map definition:

$$f : X_{\mathcal{A}} \longrightarrow \mathbb{R}^2$$
$$x \longmapsto \frac{1}{\sum_{m \in \mathcal{A}} |x_m|} \sum_{m \in \mathcal{A}} |x_m| m$$

  note that $f \circ \chi \circ H$ restricted to Δ is "almost" the same as the toric patch – *but using the $m \in \mathcal{A}$ "as the control points"*

- In some cases, we can see that $f \circ \chi \circ H$ is actually the *identity map* on Δ

An exercise:

Let Δ the triangle $\Delta_k = \mathrm{conv}\{(0,0),(k,0),(0,k)\}$ and $\mathcal{A} = \Delta_k \cap \mathbb{Z}^2$

Let w be the set of weights for the Bézier triangle toric surface patch as before.

Then for the moment map f on the degree k Veronese image, show that $f \circ \chi \circ H$ is the identity on Δ.

Linear precision

- The property $f \circ \chi \circ H = \mathrm{id}_\Delta$ is related to a property called *linear precision* in the geometric modeling world
- Often used in a less restrictive sense there, though:
- *For some weights $w_m \geq 0$ and some choice of control points m whose convex hull is Δ, the parametrized patch (with control points the m) is the identity on Δ*
- If $\mathcal{A}$ is the set of vertices of Δ and the blending functions of a patch with linear precision are *barycentric coordinates* on Δ

Final observations, continued

- Can *always reparametrize* a toric surface patch by a homeomorphism $\Delta \to \Delta$ to get linear precision
- Can also vary weights w_m and non-vertices of Δ to "tune" to obtain linear precision in some cases.
- Interesting question: which $\mathcal{A}$ and sets of blending functions on Δ have this property "automatically?" The article [BRS] shows that this true for Krasauskas' toric surface patches *only* for the Δ_k triangles, the $[p, 0] \times [0, q]$ rectangles, and certain trapezoids where X_Δ is a rational normal scroll (Hirzebruch surface)!
- One can ask the analogous question in higher dimensions too and which Δ give linear precision is an open question

Another characterization of linear precision

- Proof in [BRS] is based on the following:
- Given $\mathcal{A}$, and the w_m for $m \in \mathcal{A}$, let $f_{\mathcal{A},w} = \sum_{m \in \mathcal{A}} w_m x^m$ (a Laurent polynomial)
- Then the toric surface patch of shape $\mathcal{A}, w$ has linear precision if and only if the rational mapping $\mathbb{C}^2 \to \mathbb{C}^2$ given by

$$\frac{1}{f}\left(x_1 \frac{\partial f_{\mathcal{A},w}}{\partial x_1}, x_2 \frac{\partial f_{\mathcal{A},w}}{\partial x_2}\right)$$

(toric polar mapping) is a *birational isomorphism*

Relation to Birch's theorem

In the talk on algebraic statistics, considered toric models associated to integer matrices $\mathcal{A}$ defined by expressions like

$$\varphi : \theta \mapsto \frac{1}{\sum_{j=1}^{m} \theta^{A_j}} \left(\theta^{A_1}, \ldots, \theta^{A_m} \right).$$

The *toric model* associated to $\mathcal{A}$ is $\varphi(\mathbb{R}^d_{>0})$ and this is $(X_\mathcal{A})_{\geq 0}$ as above (abuse of notation: $\mathcal{A}$ for both the matrix and the set of lattice points; also might give a "partial" surface patch).

Given data u and $b = Au$, the MLE $\widehat{\theta}$ gives $\widehat{p} \in \varphi(\mathbb{R}^d_{>0})$ and $A\widehat{p} = \frac{1}{N}b$. Almost the same as the case where the control points are taken to be the $m \in \Delta$ (morally, $\widehat{p} \mapsto A\widehat{p}$ "is" the moment map).

References for further study

- BRS von Bothmer, Ranestad, and Sottile, Linear precision for toric surface patches, Found. of Comput. Math. 10 (2010), 37-66
- K Krasauskas, Toric Surface Patches, Adv. Comput. Math. 17 (2002), 89-113
- CGS Craciun, Garcia-Puente, and Sottile, Some geometrical aspects of control points for toric patches, Mathematical Methods for Curves and Surfaces (2010), 111-135 (arXiv 0812.1275)
- CLS Cox, Little, and Schenck, *Toric Varieties*, AMS, 2011
- F Fulton, *Introduction to toric varieties*, Princeton U. Press, 1993
- GS Garcia-Puente and Sottile, *Linear precision for parametric patches*, Adv. Comput. Math. 33 (2010), 191-214.