MONT 100N – Modeling the Environment
Solutions for Practice Problems for the Final Exam
November 30, 2017

Chapter 1

5.

(a) From the table of conversion factors, one kilometer $\doteq .6214$ mile, so one square kilometer $\doteq (.6214)^2 \doteq .3861$ square miles. Hence the Larsen C iceberg has an area of about $(.3861) \cdot 44200 \doteq 17067$ square miles.

(b) The thickness in kilometers is $350 \times .001 = .350$ kilometer. So the volume in cubic kilometers would be approximately

$$44200 \cdot .350 \doteq 15470 \text{ cubic km}$$

This is approximately

$$15470 \times (.6214)^3 = 3712$$

cubic miles. (That is a lot of ice!)

11.

(a) $\log_{10}(5.34689) \doteq .728101$. Then $\log_{10}(5.34689) \doteq 1.728101$ and $\log_{10}(53.4689) \doteq 2.728101$. What is happening here is that, for example

$$53.4689 = 5.34689 \times 10$$

so

$$\log_{10}(53.4689) = \log_{10}(5.34689) + \log_{10}(10) = 1 + \log_{10}(5.34689).$$

(b) No calculator I know of has a $\log_7$ key. So this is one place where you would need to use the conversion formula (1.2) on page 12 of the text:

$$\log_7(34.333) = \frac{\log_{10}(34.333)}{\log_{10}(7)} \doteq 1.81720.$$

(c) $\ln(100.3) \doteq 4.60817$. (For this one, you can use the ln key if your calculator has one, or use the conversion formula as in the previous problem.)

14. Since $M = \frac{2}{3}\log_{10}(S) - 10.7$, we can solve for S like this:

$$\frac{3}{2}(M + 10.7) = \log_{10}(S)$$
$$10^{\frac{3}{2}(M+10.7)} = S.$$

(a) Hence if the two quakes have moments M_1 and M_2, then

$$\frac{S_1}{S_2} = \frac{10^{\frac{3}{2}(M_1+10.7)}}{10^{\frac{3}{2}(M_2+10.7)}}$$

Subtracting the exponents on the right, we get

$$\frac{S_1}{S_2} = 10^{\frac{3}{2}(M_1 - M_2)}.$$

(b) The ratio is $10^{3/2} \doteq 31.6$. Increasing the moment magnitude by 1 gives a quake that is 31.6 times as strong in terms of the seismic moment.

Chapter 2

2.

(a) Percent change is

$$\frac{324,700,000 - 308,745,538}{308,754,538} \times 100\% \doteq 5.2\%$$

(positive since it is an increase).

(b) Percent change per year would be $\frac{5.2}{7} \doteq .743\%$ per year.

7.

(a) The densities in people per square mile are (all rounded to whole numbers):

Bronx	32900
Brooklyn	35367
Manhattan	69464
Queens	20554
StatenIsland	8030

To get the people per square kilometer multiply each by $(1.609)^2 \doteq 2.59$. The $1.609 = 1/.6124$ as in Chapter 1, problem 5 above.

(b) Not it is not. The average of the densities in part (a) would be

$$\frac{32900 + 35367 + 69464 + 20554 + 8030}{5} \doteq 33263.$$

But the actual overall NYC population density was

$$\frac{1,385,108 + 2,504,700 + 1,585,873 + 2,230,722 + 468,730}{42.1 + 70.82 + 22.83 + 108.53 + 58.37} \doteq 27016.$$

Algebraically the reason these are not the same is that you don't add fractions by adding the numerators and denominators and making a new fraction with the sums(!) In real world terms, Manhattan is over-represented in the sum of the densities because, while it has a high population density, it also has a small area. To get the city population density from the borough densities, you could multiply the borough density by the area for that borough, add, and divide by the total land area. That is the same as a weighted average where you multiply the borough density by the proportion of the total area in that borough.

Chapter 4

6.
(a) Parallel lines have equal slopes so $y - 5.2 = 7(x - 3.2)$. It's fine to leave the equation in this form, or you could also rearrange it to slope-intercept form: $y = 7x - 17.2$.
(b) Perpendicular lines have slopes that are negative reciprocals. So the equation is $y - 1 = \frac{-1}{7}(x - 0)$ or $y = \frac{-1}{7}x + 1$.

8. The only one where the slopes between successive pairs points are all the same is $g(x)$. The equation is

$$g(x) = 1.525 + -2.25(x - 1.1) = -2.25x + 4.$$

9. Using the data points $(0, 0)$ (that is, no kudzu in 1876) and $(141, 7400000)$ (that is, $7,400,000$ acres in 2017, which is 141 years after 1876), we get a linear model

$$k(t) \doteq 52482.3t$$

where $k(t)$ is the land area covered by kudzu in year t. The slope represents the growth rate of the kudzu area; the units are acres per year.

Chapter 5

2. All these equations are solved by the same technique – isolate a term with the x in the exponent on one side, take logarithms to get x out of the exponent, then solve the resulting equation for x.
(a) $(3.4)^{2x} = \frac{28.3}{4.5} \doteq 6.29$, so $2x \log_{10}(3.4) = \log_{10}(6.29)$, so

$$x \doteq \frac{\log_{10}(6.29)}{2 \log_{10}(3.4)} \doteq .7513.$$

(b) Divide by 2^x first to get $2^x = 3.5$, then $x \log_{10}(2) = \log_{10}(3.5)$ and

$$x = \frac{\log_{10}(3.5)}{\log_{10}(2)} \doteq 1.807.$$

(c) Subtract 5.6 from both sides, divide by 2.8, then take logs:

$$x = \frac{\log_{10}(2.3/2.8)}{\log_{10}(7.4)} \doteq -0.0983.$$

6.

(a) $Q(0) = 5$, so we just need to solve the equation $10 = 5(1.25)^t$ for t. The answer is $t = \frac{\log_{10}(2)}{\log_{10}(1.25)} \doteq 3.106$.
(b) Same technique as (a). The answer is .5664. (Note that $3.4 > 1.25$ means that the function here is increasing faster than the one in part (a), so the doubling time is smaller.) 7. The equation to solve is $2Q(0) = Q(0)a^t$, so $2 = a^t$ and

$$t = \frac{\log_{10}(2)}{\log_{10}(a)}$$

follows. 8. We get that when $t = t_2$, the factor $2^{\frac{t}{t_2}}$ satisfies $2^{t_2/t_2} = 2^1 = 2$. Hence $Q(t_2) = 2Q(0)$ and this exponential function does have doubling time exactly t_2. The value of a is

$$a = 2^{\frac{1}{t_2}}, \quad \text{since} \quad (2^{\frac{1}{t_2}})^t = 2^{\frac{t}{t_2}}$$

using rules for exponents.

9.
(a) The most immediate way to write down the model is

$$Q(t) = Q(0)(1/2)^{t/28.8}$$

(see equation (5.9) in the text).
(b) We solve .01% is 0.0001 so we solve

$$0.0001Q(0) = Q(0)(1/2)^{t/28.8}$$

So $t = 28.8 \cdot \frac{\log_{10}(.0001)}{\log_{10}(.5)} \doteq 382.7$ years.

Chapter 7

1. We did this in class; consult your notes. The idea is that the percent change is

$$\frac{Q(n+1) - Q(n)}{Q(n)} \times 100\% = \frac{r}{100} \times 100\%.$$

Rearrange this equation to get the one you are asked to derive.

4.
(a) $Q(n) = 3.4(1.8)^n$.
(b) Use the general solution for affine first order linear equations (equation (7.4) on page 114 of the text). The solution is

$$Q(n) = (Q(0) + \frac{b}{a-1})a^n - \frac{b}{a-1}$$

with $Q(0) = 4.3$, $a = .78$ and $b = -.03$, so

$$Q(n) = (4.3 + \frac{-.03}{-.22})(.78)^n - \frac{-.03}{-.22} \doteq 4.436(.78)^n - 4.436$$

Chapter 8

6. If you add the left, then the right sides of the SIR equations, you see

$$S(n+1) + I(n+1) + R(n+1) = S(n) + I(n) + R(n)$$

because every other term that appears in one equation with a $+$ sign occurs in a second one with a $-$ sign, so they cancel. This shows that the total population is not changing from one time step to the next.

13.
(a) $r = .03$ and $r/M = .006$, so $M = 5$. The $Q(0) < M$, so the solution will increase and approach M as a horizontal asymptote.
(b) $r = .34$ and $r/M = .0009$, so $M \doteq 378$. Since $Q(0) > M$, the solution will decrease and tend toward M as a horizontal asymptote.
(c) $= .86$ and $r/M = .0048$, so $M \doteq 179.2$. Since $Q(0) < M$, the solution will look like the one in (a).

8. (a) and (b): The modified SIR model would look like this. Say R_1 represents the infected patients who *recover* and R_2 represents the infected patients who die. Then

$$\begin{aligned} S(n+1) &= S(n) - \beta S(n)I(n) \\ I(n+1) &= I(n) + \beta S(n)I(n) - \gamma_1 I(n) - \gamma_2 I(n) \\ R_1(n+1) &= R_1(n) + \gamma_1 I(n) \\ R_2(n+1) &= R_2(n) + \gamma_2 I(n). \end{aligned}$$

As noted on the review sheet, parts (c) and (d) would not be suitable for the exam(!)

11. To find equilibrium solutions we rewrite the system (8.8) as

$$\begin{aligned} P(n+1) - P(n) &= -aP(n) + bP(n)Q(n) \\ Q(n+1) - Q(n) &= cQ(n) - cQ(n)^2/M - dP(n)Q(n) \end{aligned}$$

For an equilibrium, the right sides of both equations should be *equal to zero.* The first equation factors like this:

$$P(-a + bQ) = 0$$

so $P = 0$ or $Q = b/a$. In the second equation, if $P = 0$, then we get $Q = 0$ or $Q = M$. If $Q = b/a$, then we get $P = (1 - a/(bM)) \cdot c/d$ as desired.