MONT 100N – Modeling the Environment
Solutions/Comments on Chapter 7 Project
November 21, 2017

There was a bit of confusion about how to find and/or classify the equilibrium solutions of the difference equations representing the constant harvesting and the proportional harvesting strategies from parts B,C of the project. Here's the way you can determine them algebraically. Recall that an equilibrium solution should be a constant value E so that if $P(0) = E$, then $P(n) = E$ for all times. This means that if you compute $P(n+1) - P(n)$ at an equilibrium, then you should get 0.

B. For the constant harvesting equation:

$$P(n+1) = 1.71P(n) - .00875(P(n))^2 - h,$$

if you subtract $P(n)$ from both sides you get

$$P(n+1) - P(n) = 0.71P(n) - .00875(P(n))^2 - h.$$

Hence a population level $P = P(n)$ is an equilibrium if

$$-0.00875P^2 + 0.71P - h = 0.$$

This quadratic equation has at most two roots, given by the quadratic formula:

$$P = \frac{-0.71 \pm \sqrt{(0.71)^2 - 4(-0.00875)(-h)}}{2(-0.00875)},$$

which simplify to approximately

$$P = 40.57 \pm 0.57\sqrt{5041 - 350h}.$$

For $h = 5$, for instance, this gives two positive real values,

$$P \doteq 7.79, 73.35.$$

Now,

- The 73.35 is the one you "see" by plotting solutions of the difference equation in Excel – if $P(0)$ is large enough, then you get solutions tending to that equilibrium. *That one is a stable equilibrium, meaning that solutions that start close to that level tend toward it.*
- You don't see the 7.79 so directly. However, that is actually the "cutoff" value between the initial values $P(0)$ that produce solutions tending up to 73.35, and the other solutions that go to 0 and then become negative (the ones where the population "crashes"). *That one is an unstable equilibrium, meaning that unless $P(0)$ equals the*

actual root of the quadratic equation above, then the solution will tend away from that equilbrium.

You should have noticed that the harvesting level $h = 20$ yields *only solutions that "crash"* no matter what $P(0)$ is. The reason for this is that when $h = 20$, the roots of the quadratic equation above are actually non-real because $5041 - 350 \cdot 20 < 0$ under the square root. In fact, the largest value of h for which there are real roots is the solution of

$$5041 - 350h = 0 \Rightarrow h \doteq 14.4$$

This is (about) the largest h for which a stable equilibrium exists. If you harvest at this constant level, and $P(0)$ is large enough (say > 40.57), then you can keep doing this forever, so the strategy is *sustainable* in that sense. The equilibrium population level in this case is about 40.57.

C. The proportional harvesting equation is

$$P(n+1) - P(n) = 0.71P(n) - 0.00875(P(n))^2 - pP(n),$$

or

$$P(n+1) - P(n) = (0.71 - p) \cdot P(n) - 0.00875(P(n))^2.$$

The equilibrium values are determined by setting the right side equal to zero, just as in question B. This gives equilibrium levels at

$$P = 0, P = \frac{0.71 - p}{0.00875}.$$

Note that if $p > .71$, then the second one becomes negative and hence unrealistic as a population value.

If $p = 0.3$, then we get a positive equilibrium at $P = \frac{.71-0.3}{0.00875} \doteq 46.86$. This can be seen to be stable by looking at the solutions in Excel. If we harvest over a long period then we will eventually tend to $(.3) \cdot 46.86 \doteq 14.1$ million kg of fish harvested per year.

D. To compare the three harvesting methods, we might first want to take into account whether the method is realistic. That rules out A pretty quickly. After all, who would want to eat halibut that has been sitting around in a freezer for up to 11 years?? Between B and C, the choice could be made on which one would produce the largest yearly take. We saw above that we cannot do better than 14.4 million kg per year on a sustainable basis using constant harvesting. The constant level $h = 14.4$ is also problematic because any $P(0)$ less than 40.57 leads to a crash, and that level is sustainable only if $P(0) > 40.57$. With proportional harvesting, though if $p = .355$ we can take the same amount of fish per year (14.4×10^6 kg) but the equilbrium level population level $\frac{.71-.355}{0.00875} = 40.57$ is still a stable equilibrium. In other words, "proportional harvesting is the way to go."