The trigonometric substitution method handles many integrals containing expressions like
\[ \sqrt{a^2 - x^2}, \sqrt{x^2 + a^2}, \sqrt{x^2 - a^2} \]
(possibly including expressions without the square roots!) The basis for this approach is the trigonometric identities
\[
1 = \sin^2 \theta + \cos^2 \theta \\
\Rightarrow \sec^2 \theta = \tan^2 \theta + 1.
\]
from which we derive other related identities:
\[
\sqrt{a^2 - (a \sin \theta)^2} = a \cos \theta \\
\sqrt{(a \tan \theta)^2 + a^2} = a \sec \theta \\
\sqrt{(a \sec \theta)^2 - a^2} = a \tan \theta
\]
Hence,

1. If our integral contains \( \sqrt{a^2 - x^2} \), the substitution \( x = a \sin \theta \) will convert this radical to the simpler form \( a \cos \theta \).
2. If our integral contains \( \sqrt{x^2 + a^2} \), the substitution \( x = a \tan \theta \) will convert this radical to the simpler form \( a \sec \theta \).
3. If our integral contains \( \sqrt{x^2 - a^2} \), the substitution \( x = a \sec \theta \) will convert this radical to the simpler form \( a \tan \theta \).

Then we substitute for the rest of the integral, integrate the resulting trigonometric form, and convert back to the original variable.

Two Examples

A) Compute \( \int \frac{u^2}{\sqrt{a^2 - u^2}} \, du \). Solution: The \( \sqrt{a^2 - u^2} \) tells us that we want the sine substitution: \( u = a \sin \theta \). Then \( du = a \cos \theta \, d\theta \), and the integral becomes:
\[
\int \frac{a^3 \sin^2 \theta \cos \theta}{a \cos \theta} \, d\theta = a^2 \int \sin^2 \theta \, d\theta
\]
We apply the half-angle formula \( \sin^2 \theta = \frac{1}{2}(1 - \cos(2\theta)) \) to this and integrate
\[
a^2 \int \frac{1}{2}(1 - \cos(2\theta)) \, d\theta
\]
to yield
\[ \frac{a^2}{2} \theta - \frac{a^2}{4} \sin(2\theta) = \frac{a^2}{2} \theta - \frac{a^2}{2} \sin \theta \cos \theta + C. \]

Then, we convert back to functions of \( u \) using the substitution equation \( u = a \sin \theta \). From this,
\[ \theta = \arcsin(u/a), \quad \cos \theta = \sqrt{a^2 - u^2}/a, \quad \sin \theta = u/a \]
so the integral is
\[ \int \frac{u^2}{\sqrt{a^2 - u^2}} \, du = \frac{a^2}{2} \arcsin(u/a) - \frac{1}{2} u \sqrt{a^2 - u^2} + C. \]

B) Compute \( \int \frac{dx}{x\sqrt{x^2+16}} \). Solution: The \( \sqrt{x^2+16} \) indicates that we want the tangent substitution \( x = 4 \tan \theta \). Then \( dx = 4 \sec^2 \theta \, d\theta \) and the integral becomes:
\[ \int \frac{4 \sec^2 \theta \, d\theta}{4 \tan \theta \cdot 4 \sec \theta} = \frac{1}{4} \int \frac{\sec \theta \, d\theta}{\tan \theta} = \frac{1}{4} \int \frac{1}{\sin \theta} \, d\theta \]
Since \( 1/\sin(\theta) = \csc(\theta) \), from \# 15 in Stewart’s table of integrals,
\[ \frac{1}{4} \int \csc(\theta) \, d\theta = \frac{1}{4} \ln |\csc(\theta) - \cot(\theta)| + C \]
Then, from \( x = 4 \tan \theta \), we get \( \cos \theta = \frac{4}{\sqrt{x^2+16}} \) and \( \sin \theta = \frac{x}{\sqrt{x^2+16}} \). Hence
\[ \csc(\theta) = \frac{\sqrt{x^2+16}}{x} \quad \text{and} \quad \cot(\theta) = \frac{4}{x}. \]
The integral equals:
\[ \frac{1}{4} \ln \left| \frac{\sqrt{x^2+16}}{x} - \frac{4}{x} \right| + C. \]