Mathematics 132 – Calculus for Physical and Life Sciences, 2
Summary of Trigonometric Substitutions
Spring 2008

The trigonometric substitution method handles many integrals containing expressions like

$$\sqrt{a^2 - x^2}, \sqrt{x^2 + a^2}, \sqrt{x^2 - a^2}$$

(possibly including expressions without the square roots!) The basis for this approach is the trigonometric identities

$$1 = \sin^2\theta + \cos^2\theta$$
$$\Rightarrow \sec^2\theta = \tan^2\theta + 1.$$

from which we derive other related identities:

$$\sqrt{a^2 - (a\sin\theta)^2} = a\cos\theta$$
$$\sqrt{(a\tan\theta)^2 + a^2} = a\sec\theta$$
$$\sqrt{(a\sec\theta)^2 - a^2} = a\tan\theta$$

Hence,

1. If our integral contains $\sqrt{a^2 - x^2}$, the substitution $x = a\sin\theta$ will convert this radical to the simpler form $a\cos\theta$.
2. If our integral contains $\sqrt{x^2 + a^2}$, the substitution $x = a\tan\theta$ will convert this radical to the simpler form $a\sec\theta$.
3. If our integral contains $\sqrt{x^2 - a^2}$, the substitution $x = a\sec\theta$ will convert this radical to the simpler form $a\tan\theta$.

Then we substitute for the rest of the integral, integrate the resulting trigonometric form, and convert back to the original variable.

Two Examples

A) Compute $\int \frac{u^2}{\sqrt{a^2-u^2}}\,du$. Solution: The $\sqrt{a^2 - u^2}$ tells us that we want the sine substitution: $u = a\sin\theta$. Then $du = a\cos\theta\,d\theta$, and the integral becomes:

$$\int \frac{a^3\sin^2\theta\cos\theta\,d\theta}{a\cos\theta} = a^2\int \sin^2\theta\,d\theta$$

We apply the half-angle formula $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$ to this and integrate

$$a^2\int \frac{1}{2}(1 - \cos(2\theta))\,d\theta$$

to yield

$$= \frac{a^2}{2}\theta - \frac{a^2}{4}\sin(2\theta) = \frac{a^2\theta}{2} - \frac{a^2}{2}\sin\theta\cos\theta + C.$$

Then, we convert back to functions of u using the substitution equation $u = a\sin\theta$. From this,

$$\theta = \arcsin(u/a), \qquad \cos\theta = \sqrt{a^2-u^2}/a, \qquad \sin\theta = u/a$$

so the integral is

$$\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = \frac{a^2}{2}\arcsin(u/a) - \frac{1}{2}u\sqrt{a^2-u^2} + +C.$$

B) Compute $\int \frac{dx}{x\sqrt{x^2+16}}$. Solution: The $\sqrt{x^2+16}$ indicates that we want the tangent substitution $x = 4\tan\theta$. Then $dx = 4\sec^2\theta\ d\theta$ and the integral becomes:

$$\int \frac{4\sec^2\theta\ d\theta}{4\tan\theta\cdot 4\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan\theta}\ d\theta = \frac{1}{4}\int \frac{1}{\sin\theta}\ d\theta$$

Since $1/\sin(\theta) = \csc(\theta)$, from # 15 in Stewart's table of integrals,

$$\frac{1}{4}\int \csc(\theta)\ d\theta = \frac{1}{4}\ln|\csc(\theta) - \cot(\theta)| + C$$

Then, from $x = 4\tan\theta$, we get $\cos\theta = \frac{4}{\sqrt{x^2+16}}$ and $\sin\theta = \frac{x}{\sqrt{x^2+16}}$. Hence

$$\csc(\theta) = \frac{\sqrt{x^2+16}}{x} \quad \text{and} \quad \cot(\theta) = \frac{4}{x}.$$

The integral equals:

$$\frac{1}{4}\ln\left|\frac{\sqrt{x^2+16}}{x} - \frac{4}{x}\right| + C.$$