The \textit{partial fraction} method applies to rational functions

\[
h(x) = \frac{f(x)}{g(x)} = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}
\]

(or to functions that can be brought to this form by a preliminary substitution). The steps involved are:

1. If \(n \geq m\), first divide \(g(x)\) into \(f(x)\) using polynomial division to write \(f(x) = q(x)g(x) + r(x)\), where the degree of \(r(x)\) is less than \(m\). This yields

\[
\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}
\]

and

\[
\int \frac{f(x)}{g(x)} \, dx = \int q(x) \, dx + \int \frac{r(x)}{g(x)} \, dx
\]

2. Now, assuming we have a rational function where the degree of the numerator is strictly less than the degree of the denominator, \textit{factor} the denominator completely into linear and quadratic factors. The quadratic factors will have no real roots when they cannot be factored further.

3. Set up the partial fractions. If \((x + a)^c\) is the highest power of a linear polynomial that divides \(g(x)\), then the partial fractions will include a group of terms

\[
\frac{A_1}{x + a} + \frac{A_2}{(x + a)^2} + \cdots + \frac{A_c}{(x + a)^c}
\]

If \((ax^2 + bx + c)^f\) is the highest power of a quadratic with no real roots that divides \(g(x)\), then the partial fractions will include a group of terms

\[
\frac{B_1 x + C_1}{ax^2 + bx + c} + \frac{B_2 x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_f x + C_f}{(ax^2 + bx + c)^f}.
\]

4. Solve for the coefficients in the partial fractions. This can be done by clearing denominators, then substituting in \(x\)-values and/or equating coefficients on both sides of the resulting equation.

5. Integrate the partial fractions.

Here is an example illustrating many of these steps. Suppose we need to integrate

\[
\int \frac{x^5 + 4x + 1}{x^4 + 9x^2} \, dx.
\]
The degree of the top is larger, so we divide first:

\[ x^5 + 4x + 1 = x(x^4 + 9x^2) + (-9x^3 + 4x + 1) \]

so

\[ \frac{x^5 + 4x + 1}{x^4 + 9x^2} = x + \frac{-9x^3 + 4x + 1}{x^4 + 9x^2} \]

The denominator factors as \( x^4 + 9x^2 = x^2(x^2 + 9) \). So the partial fractions are

\[ \frac{-9x^3 + 4x + 1}{x^4 + 9x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 9}. \]

Clearing denominators,

\[ -9x^3 + 4x + 1 = Ax(x^2 + 9) + B(x^2 + 9) + (Cx + D)x^2. \]

Substituting \( x = 0 \) we see \( 1 = 9B \) so \( B = 1/9 \). From the coefficient of \( x \) we see \( 4 = 9A \), so \( A = 4/9 \). Then from the coefficients of \( x^3 \), \( A + C = -9 \), so \( C = -9 - 4/9 = -85/9 \) and finally from the coefficient of \( x^2 \), \( 0 = B + D \), so \( D = -1/9 \). This gives

\[
\int \frac{x^5 + 4x + 1}{x^4 + 9x^2} \, dx = \int x + \frac{-9x^3 + 4x + 1}{x^4 + 9x^2} \, dx \\
= \int \left( x + \frac{4/9}{x} + \frac{1/9}{x^2} + \frac{(-85/9)x + (-1/9)}{x^2 + 9} \right) \, dx \\
= \frac{x^2}{2} + \frac{4}{9} \ln |x| - \frac{1}{9} \frac{1}{x} - \frac{85}{18} \ln(x^2 + 9) - \frac{1}{27} \tan^{-1}(x/3) + C.
\]