Mathematics 132 – Calculus for Physical and Life Sciences 2
Integration by Partial Fractions
February 11 - 12, 2008

The *partial fraction* method applies to rational functions

$$h(x) = \frac{f(x)}{g(x)} = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$$

(or to functions that can be brought to this form by a preliminary substitution). The steps involved are:

1. If $n \geq m$, first *divide* $g(x)$ into $f(x)$ using polynomial division to write $f(x) = q(x)g(x) + r(x)$, where the degree of $r(x)$ is less than m. This yields

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

and

$$\int \frac{f(x)}{g(x)}\,dx = \int q(x)\,dx + \int \frac{r(x)}{g(x)}\,dx$$

2. Now, assuming we have a rational function where the degree of the numerator is strictly less than the degree of the denominator, *factor* the denominator completely into linear and quadratic factors. The quadratic factors will have no real roots when they cannot be factored further.
3. Set up the partial fractions. If $(x+a)^e$ is the highest power of a linear polynomial that divides $g(x)$, then the partial fractions will include a group of terms

$$\frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \cdots + \frac{A_e}{(x+a)^e}$$

If $(ax^2+bx+c)^f$ is the highest power of a quadratic with no real roots that divides $g(x)$, then the partial fractions will include a group of terms

$$\frac{B_1 x + C_1}{ax^2+bx+c} + \frac{B_2 x + C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_f x + C_f}{(ax^2+bx+c)^f}.$$

4. Solve for the coefficients in the partial fractions. This can be done by clearing denominators, then substituting in x-values and/or equating coefficients on both sides of the resulting equation.
5. Integrate the partial fractions.

Here is an example illustrating many of these steps. Suppose we need to integrate

$$\int \frac{x^5 + 4x + 1}{x^4 + 9x^2}\,dx.$$

The degree of the top is larger, so we divide first:

$$x^5 + 4x + 1 = x(x^4 + 9x^2) + (-9x^3 + 4x + 1)$$

so

$$\frac{x^5 + 4x + 1}{x^4 + 9x^2} = x + \frac{-9x^3 + 4x + 1}{x^4 + 9x^2}$$

The denominator factors as $x^4 + 9x^2 = x^2(x^2 + 9)$. So the partial fractions are

$$\frac{-9x^3 + 4x + 1}{x^4 + 9x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 9}.$$

Clearing denominators,

$$-9x^3 + 4x + 1 = Ax(x^2 + 9) + B(x^2 + 9) + (Cx + D)x^2.$$

Substituting $x = 0$ we see $1 = 9B$ so $B = 1/9$. From the coefficient of x we see $4 = 9A$, so $A = 4/9$. Then from the coefficients of x^3, $A + C = -9$, so $C = -9 - 4/9 = -85/9$ and finally from the coefficient of x^2, $0 = B + D$, so $D = -1/9$. This gives

$$\begin{aligned}
\int \frac{x^5 + 4x + 1}{x^4 + 9x^2}\,dx &= \int x + \frac{-9x^3 + 4x + 1}{x^4 + 9x^2}\,dx \\
&= \int x + \frac{4/9}{x} + \frac{1/9}{x^2} + \frac{(-85/9)x + (-1/9)}{x^2 + 9}\,dx \\
&= \frac{x^2}{2} + \frac{4}{9}\ln|x| - \frac{1}{9}\frac{1}{x} - \frac{85}{18}\ln(x^2 + 9) - \frac{1}{27}\tan^{-1}(x/3) + C.
\end{aligned}$$