Solutions for Problem Set 2

MATH 131, Fall 2007

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Section 1.5

4. The graphs \( y = e^x \) and \( y = 8^x \) are increasing exponentials; \( y = e^{-x} \) is the reflection of \( y = e^x \) across the \( y \)-axis and \( y = 8^{-x} \) is the reflection of \( y = 8^x \) across the \( y \)-axis. Those two are decreasing exponentials.

8. Shift \( y = 4^x \) 3 units to the right (\( y \)-axis intercept becomes \( (0, 4^{-3}) = (0, \frac{1}{64}) \)).

10. Take \( y = e^x \), stretch vertically by a factor of 2, then shift up one unit.

13.
   a. \( y = e^x - 2 \)
   b. \( y = e^{x-2} \)
   c. \( y = -e^x \)
   d. \( y = e^{-x} \)
   e. \( y = -e^{-x} \)

14.
   a. \( y = 8 - e^x \) (same as reflecting across the \( x \)-axis, then shifting up by 8 units).
   b. \( y = e^{4-x} = e^{-(x-4)} \) (same as shifting right by 4 units, then reflect across the \( y \)-axis).
17. From the given points, $6 = Ca$ and $24 = Ca^3$. Hence $4 = \frac{Ca^3}{Ca} = a^2$, so $a = 2$ and $C = 3$. $f(x) = 3 \cdot 2^x$.

18. Same method as 17: $2 = Ca = C$ and $2/9 = Ca^2 = 2a^2$. So $a^2 = 1/9$, and $a = 1/3$. $f(x) = 2 \cdot \left(\frac{1}{3}\right)^x$.

20. Definitely take option II. The reason is that since each month has at least 28 days, on the last day you will get at least $2^{27}$ pennies. This is $(0.01)134,217,728 = $1,342,177.28, which is more than a million dollars, just on the last day.

26.

a. There are 6 doublings in the first three hours, so $500 \cdot 2^6 = 32000$.

b. After $t$ hours, there are $500 \cdot 2^t$ bacteria.

c. 40 minutes $= \frac{2}{3}$ hour, so $500 \cdot 2^{4/3} \simeq 1260$ bacteria.

d. The graph is a standard increasing exponential, with vertical axis intercept at 500. The population reaches 10000 when $100000 = 500 \cdot 2^t$, so $200 = 2^2$, so $t = \frac{1}{2} \log_2(200)$ hours (about 3.82 hours).

Section 1.6

4. Apparently one-to-one, since no duplicate $f(x)$-values appear in table.

6. Not one-to-one. Note that $f(x) = 0$ for three different $x$-values in the picture. The graph does not pass the horizontal line test.

10. The graph $y = 10 - 3x$ is a line with slope $-3$, so it crosses horizontal lines only once. This is a one-to-one function.

12. Not one-to-one, since, for instance $\cos(0) = 1 = \cos(2\pi)$

14. Not one-to-one, since peoples' heights typically stay roughly constant for extended periods (e.g. between growth spurts as children, or once they reach maturity).

15. $f^{-1}(9) = 2$. 

2
16 b. For any function \( f \) that is one-to-one, if 5 is in the range of \( f \), 
\( f(f^{-1}(5)) = 5 \). (This has nothing to do with the given formula for \( f(x) \).

21. \( y = \sqrt{10 - 3x} \) implies \( x = \frac{1}{3}(y^2 - 10) = \frac{10}{3} - \frac{y^2}{3} \). So \( f^{-1}(x) = \frac{10}{3} - \frac{x^2}{3} \).

24. \( y = 2x^3 + 3 \) implies \( x = \left(\frac{y-3}{2}\right)^{1/3} = \frac{3}{2} \sqrt[3]{\frac{y-3}{2}} \), so \( f^{-1}(x) = \frac{3}{2} x^{1/3} \).

26. \( y = \frac{1+e^x}{1-e^x} \) implies \( y - ye^x = 1 + e^x \), so \( e^x = \frac{y-1}{y+1} \), and \( x = \ln \left( \frac{y-1}{y+1} \right) \). The inverse function is \( f^{-1}(x) = \ln \left( \frac{y-1}{y+1} \right) \).

30. Reflect the graph across the line \( y = x \). The inverse function graph should be decreasing from about \((-1, 4)\), through \((0, 2)\), to \((1, 0)\).

34. a. \( \log_8(2) = 1/3 \), since \( 8^{1/3} = 2 \). b. \( \ln(e^{\sqrt{2}}) = \sqrt{2} \).

36. a) Using properties of logs and exponentials, \( 2^{\log_2(3)+\log_2(5)} = 2^{\log_2(15)} = 15 \).

48.

a. Take \( \ln \) of both sides in \( e^{2x+3} = 7 \). We get \( 2x + 3 = \ln(7) \), so \( x = \frac{\ln(7) - 3}{2} \).

b. Take exponential of both sides in \( \ln(5 - 2x) = -3 \) to get \( 5 - 2x = e^{-3} \).
Then \( x = \frac{5 - e^{-3}}{2} \).

50 a. Take exponentials twice, \( \ln(\ln(x)) = 1 \) implies \( \ln(x) = e \), so \( x = e^e \).

54. Let \( f(x) = \ln(2 + \ln(x)) \). The domain of this composition is the set of \( x \) in the domain of \( \ln(x) \) such that \( 2 + \ln(x) \) is in the domain of the “outside” \( \ln \). Recall that \( \ln(x) \) has domain the set of all positive reals. But \( 2 + \ln(x) > 0 \) only for \( x > e^{-2} \). So the domain is the set of all real \( x \) with \( x > e^{-2} \).
The inverse function is found by the usual process: \( y = \ln(2 + \ln(x)) \) when \( e^y = 2 + \ln(x) \), so \( e^{e^{y-2}} = x \). Thus \( f^{-1}(x) = e^{e^{y-2}} \). The domain of this function is the same as the range of \( f(x) \), which is the set of all real \( x \).
Appendix C

2. a. $-315^\circ = -\frac{7\pi}{4}$ radians. (The negative angle means measure clockwise, not counterclockwise as usual.) b. $36^\circ = \frac{\pi}{5}$ radians.

4. a. $-\frac{7\pi}{2} = -630^\circ$. b. $\frac{8\pi}{3} = 480^\circ$.

16. $x = 25\cos(40^\circ) \approx 19.15111$ (using Maple).

24. Square out the left:

$$(\sin(x) + \cos(x))^2 = \sin^2(x) + 2\sin(x)\cos(x) + \cos^2(x).$$

Regrouping the terms, we have

$$= (\sin^2(x) + \cos^2(x)) + 2\sin(x)\cos(x) = 1 + \sin(2x),$$

by the basic identity 7, and the double angle formula 15a.

30. $2\sin^2(x) = 1$ when $\sin(x) = \pm\frac{1}{\sqrt{2}}$, so $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (the first two give $+\frac{1}{\sqrt{2}}$ the other two give $-\frac{1}{\sqrt{2}}$).

32. $|\tan(x)| = 1$ when $\tan(x) = \pm 1$, so $\sin(x) = \pm \cos(x)$. This happens at the same $x$-values given in problem 30: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

37. Shift the graph $y = \cos(x)$ by $\frac{\pi}{3}$ units to the right.

38. Compress the graph $y = \tan(x)$ by a factor of 2 horizontally. This puts vertical asymptotes at $x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$, etc.

41. a. This is the unique angle $\theta$ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\sin(\theta) = .5$, so $\theta = \frac{\pi}{6}$. b. $\arctan(-1) = \tan^{-1}(-1) = -\frac{\pi}{4}$.

42. a. $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$. b. $\arcsin(1) = \sin^{-1}(1) = \frac{\pi}{2}$.

43. a. $\sin(\sin^{-1}(0.7)) = 0.7$ (see problem 16 b in Section 1.6 above). b. $\sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}$, so $\arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$. (Values of $\arcsin = \sin^{-1}$ are all in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.)
44. a) If $\theta = \arctan(2)$, then $\tan(\theta) = 2$, so we can get $\theta$ in a right triangle with opposite side 2 and adjacent side 1 (draw the picture!). Then the hypotenuse is $\sqrt{5}$, so $\sec(\theta) = \sqrt{5}$ (hypotenuse/adjacent).

   b) First use the double angle formula:

   \[
   \sin(2 \sin^{-1}(3/5)) = 2\sin(\sin^{-1}(3/5))\cos(\sin^{-1}(3/5)) = 2 \cdot \frac{3}{5} \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{24}{25}.
   \]

   Note: the value $\frac{4}{5} = \cos(\sin^{-1}(3/5))$ is obtained by a calculation similar to what we did in part a. If $\theta = \sin^{-1}(3/5)$, then $\theta$ is an angle in a right triangle with opposite side 3 and hypotenuse 5. The adjacent side is $\sqrt{25 - 9} = 4$, so $\cos(\theta) = 4/5$ (adjacent/hypotenuse).