Solutions for Problem Set 2

MATH 131, Fall 2007

September 17, 2007

Section 1.5

4. The graphs $y = e^x$ and $y = 8^x$ are increasing exponentials; $y = e^{-x}$ is the reflection of $y = e^x$ across the y-axis and $y = 8^{-x}$ is the reflection of $y = 8^x$ across the y-axis. Those two are decreasing exponentials.

8. Shift $y = 4^x$ 3 units to the right (y-axis intercept becomes $(0, 4^{-3}) = \left(0, \frac{1}{64}\right)$).

10. Take $y = e^x$, stretch vertically by a factor of 2, then shift up one unit.

13.

a. $y = e^x - 2$

b. $y = e^{x-2}$

c. $y = -e^x$

d. $y = e^{-x}$

e. $y = -e^{-x}$

14.

a. $y = 8 - e^x$ (same as reflecting across the x-axis, then shifting up by 8 units).

b. $y = e^{4-x} = e^{-(x-4)}$ (same as shifting right by 4 units, then reflect across the y-axis).

17. From the given points, $6 = Ca$ and $24 = Ca^3$. Hence $4 = \frac{Ca^3}{Ca} = a^2$, so $a = 2$ and $C = 3$. $f(x) = 3 \cdot 2^x$.

18. Same method as 17: $2 = Ca^0 = C$ and $2/9 = Ca^2 = 2a^2$. So $a^2 = 1/9$, and $a = 1/3$. $f(x) = 2 \cdot \left(\frac{1}{3}\right)^x$.

20. Definitely take option II. The reason is that since each month has at least 28 days, on the last day you will get at least 2^{27} pennies. This is $\$(.01)134,217,728 = \$1,342,177.28$, which is more than a million dollars, just on the last day.

26.

a. There are 6 doublings in the first three hours, so $500 \cdot 2^6 = 32000$.

b. After t hours, there are $500 \cdot 2^{2t}$ bacteria.

c. 40 minutes $= \frac{2}{3}$ hour, so $500 \cdot 2^{4/3} \doteq 1260$ bacteria.

d. The graph is a standard increasing exponential, with vertical axis intercept at 500. The population reaches 10000 when $100000 = 500 \cdot 2^{2t}$, so $200 = 2^{2t}$, so $t = \frac{1}{2}\log_2(200)$ hours (about 3.82 hours).

Section 1.6

4. Apparently one-to-one, since no duplicate $f(x)$-values appear in table.

6. Not one-to-one. Note that $f(x) = 0$ for three different x-values in the picture. The graph does not pass the horizontal line test.

10. The graph $y = 10 - 3x$ is a line with slope -3, so it crosses horizontal lines only once. This is a one-to-one function.

12. Not one-to-one, since, for instance $\cos(0) = 1 = \cos(2\pi)$

14. Not one-to-one, since peoples' heights typically stay roughly constant for extended periods (e.g. between growth spurts as children, or once they reach maturity).

15. $f^{-1}(9) = 2$.

16 b. For any function f that is one-to-one, if 5 is in the range of f, $f(f^{-1}(5)) = 5$. (This has nothing to do with the given formula for $f(x)$.)

21. $y = \sqrt{10-3x}$ implies $x = \frac{1}{-3}(y^2-10) = \frac{10}{3} - \frac{y^2}{3}$. So $f^{-1}(x) = \frac{10}{3} - \frac{x^2}{3}$.

24. $y = 2x^3 + 3$ implies $x = \left(\frac{y-3}{2}\right)^{1/3} = \sqrt[3]{\frac{y-3}{2}}$, so $f^{-1}(x) = \sqrt[3]{\frac{x-3}{2}}$.

26. $y = \frac{1+e^x}{1-e^x}$ implies $y - ye^x = 1 + e^x$, so $e^x = \frac{y-1}{y+1}$, and $x = \ln\left(\frac{y-1}{y+1}\right)$. The inverse function is $f^{-1}(x) = \ln\left(\frac{x-1}{x+1}\right)$.

30. Reflect the graph across the line $y = x$. The inverse function graph should be decreasing from about $(-1,4)$, through $(0,2)$, to $(1,0)$.

34. a. $\log_8(2) = 1/3$, since $8^{1/3} = 2$. b. $\ln(e^{\sqrt{2}}) = \sqrt{2}$.

36. a) Using properties of logs and exponentials, $2^{\log_2(3)+\log_2(5)} = 2^{\log_2(15)} = 15$.

48.

a. Take ln of both sides in $e^{2x+3} = 7$. We get $2x+3 = \ln(7)$, so $x = \frac{\ln(7)-3}{2}$.

b. Take exponential of both sides in $\ln(5-2x) = -3$ to get $5 - 2x = e^{-3}$. Then $x = \frac{5-e^{-3}}{2}$.

50 a. Take exponentials twice, $\ln(\ln(x)) = 1$ implies $\ln(x) = e$, so $x = e^e$.

54. Let $f(x) = \ln(2+\ln(x))$. The domain of this composition is the set of x in the domain of $\ln(x)$ such that $2+\ln(x)$ is in the domain of the "outside" ln. Recall that $\ln(x)$ has domain the set of all positive reals. But $2+\ln(x) > 0$ only for $x > e^{-2}$. So the domain is the set of all real x with $x > e^{-2}$. The inverse function is found by the usual process: $y = \ln(2+\ln(x))$ when $e^y = 2 + \ln(x)$, so $e^{e^y-2} = x$. Thus $f^{-1}(x) = e^{e^x-2}$. The domain of this function is the same as the range of $f(x)$, which is the set of all real x.

Appendix C

2. a. $-315° = -\frac{7\pi}{4}$ radians. (The negative angle means measure clockwise, not counterclockwise as usual.) b. $36° = \frac{\pi}{5}$ radians.

4. a. $-\frac{7\pi}{2} = -630°$. b. $\frac{8\pi}{3} = 480°$.

16. $x = 25\cos(40°) \doteq 19.15111$ (using Maple).

24. Square out the left:

$$(\sin(x) + \cos(x))^2 = \sin^2(x) + 2\sin(x)\cos(x) + \cos^2(x).$$

Regrouping the terms, we have

$$= (\sin^2(x) + \cos^2(x)) + 2\sin(x)\cos(x) = 1 + \sin(2x),$$

by the basic identity 7, and the double angle formula 15a.

30. $2\sin^2(x) = 1$ when $\sin(x) = \pm\frac{1}{\sqrt{2}}$, so $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (the first two give $+\frac{1}{\sqrt{2}}$ the other two give $-\frac{1}{\sqrt{2}}$).

32. $|\tan(x)| = 1$ when $\tan(x) = \pm 1$, so $\sin(x) = \pm\cos(x)$. This happens at the same x-values given in problem 30: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

37. Shift the graph $y = \cos(x)$ by $\frac{\pi}{3}$ units to the right.

38. Compress the graph $y = \tan(x)$ by a factor of 2 horizontally. This puts vertical asymptotes at $x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$, etc.

41. a. This is the unique angle θ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin(\theta) = .5$, so $\theta = \frac{\pi}{6}$. b. $\arctan(-1) = \tan^{-1}(-1) = -\frac{\pi}{4}$.

42. a. $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$. b. $\arcsin(1) = \sin^{-1}(1) = \frac{\pi}{2}$.

43. a. $\sin(\sin^{-1}(0.7)) = 0.7$ (see problem 16 b in Section 1.6 above). b. $\sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}$, so $\arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$. (Values of $\arcsin = \sin^{-1}$ are all in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.)

44. a) If $\theta = \arctan(2)$, then $\tan(\theta) = 2$, so we can get θ in a right triangle with opposite side 2 and adjacent side 1 (draw the picture!). Then the hypotenuse is $\sqrt{5}$, so $\sec(\theta) = \sqrt{5}$ (hypotenuse/adjacent).

b) First use the double angle formula:

$$\sin(2\sin^{-1}(3/5)) = 2\sin(\sin^{-1}(3/5))\cos(\sin^{-1}(3/5)) = 2\cdot\frac{3}{5}\cdot\sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{24}{25}.$$

Note: the value $\frac{4}{5} = \cos(\sin^{-1}(3/5))$ is obtained by a calculation similar to what we did in part a. If $\theta = \sin^{-1}(3/5)$, then θ is an angle in a right triangle with opposite side 3 and hypotenuse 5. The adjacent side is $\sqrt{25-9} = 4$, so $\cos(\theta) = 4/5$ (adjacent/hypotenuse).