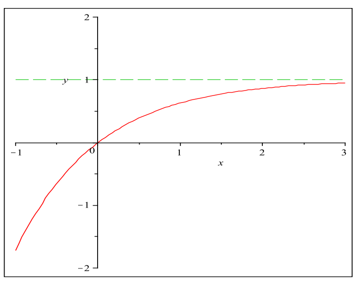
College of the Holy Cross, Fall 2007
Math 131, Midterm 1 (All Sections)
Wednesday, September 19, 6 PM
Solutions

I. Match the plots below with the following formulas. Note that there is an extra graph.

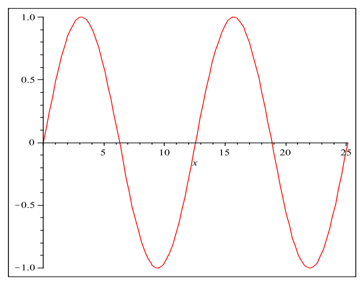
(4) A) $y = 3 - (x-1)^2$ is Plot: 3 (the graph of a quadratic equation is a parabola and the point $(1, 3)$ is on the graph.)

(4) B) $y = \sin(x/2)$ is Plot: 2 (the graph of $y = \sin(x)$ is stretched horizontally by a factor of 2.)

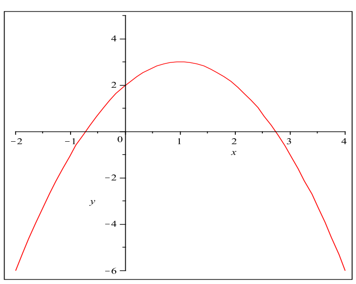
(4) C) $y = 1 - e^{-x}$ is Plot: 1 (the graph $y = e^x$ is reflected about the y-axis, then about the x-axis and finally it is shifted up one unit.)



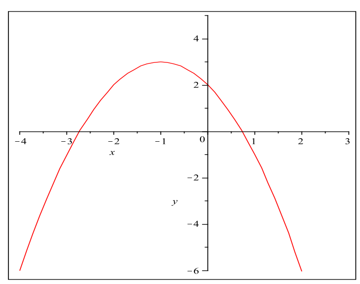
Plot 1:



Plot 2:



Plot 3:



Plot 4:

II. The manager of a furniture factory has collected the following data for the cost of manufacturing chairs.

# Chairs (per day)	Cost (in dollars)
100	2400
150	3100
250	4500
300	5200

(6) A) Express the cost (y) as a function of the number of chairs produced per day (x), assuming that it is linear.

Solution: Let $C(N)$ represent the cost when N chairs are produced per day. Since the function is linear, it is of the form $C(N) = mN + b$.

The slope is $m = \dfrac{3100 - 2400}{150 - 100} = \dfrac{700}{50} = 14$. Then $2400 = 14 \cdot 100 + b$, so $b = 1000$. Therefore $C(N) = 14N + 1000$.

(3) B) What is the slope, and what does it represent in real-world terms?

Solution: The slope is equal to 14 and is represents the cost of producing an additional chair per day.

(3) C) What is the y-intercept, and what does it represent in terms of cost?

Solution: The y-intercept is 1000 and it represents the overhead cost (equipment maintenance, facility rent, insurance, etc.)

(3) D) Using your model, determine how much it will cost to produce 350 chairs per day.

Solution: $C(350) = 14 \cdot 350 + 1000 = 5900$

III. Given $f(x) = 4 - x^2$ and $g(x) = \sqrt{3x-2}$, answer the following questions.

(6) A) Find the domain of $f(x)$ and the domain of $g(x)$.

Solution: The domain of f is the set of all real numbers.

The domain of g is the set of real numbers satisfying $3x - 2 \geq 0$, i.e., $x \geq \frac{2}{3}$. Thus the domain of g is the interval $[2/3, \infty)$.

(6) B) Find the function f/g and its domain.

Solution: $\left(\frac{f}{g}\right)(x) = \frac{4-x^2}{\sqrt{3x-2}}$ and its domain is $(2/3, \infty)$.

(6) C) Find the function $f \circ g$ and its domain.

Solution: $(f \circ g)(x) = f(g(x)) = 4 - (\sqrt{3x-2})^2 = 4 - (3x-2) = 6 - 3x$.

Its domain is the domain of g: $[2/3, \infty)$.

IV. Answer the following questions.

(5) A) Find all values of x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which $|\tan x| > 1$.

Solution: We need all values of x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan x < -1$ or $\tan x > 1$. Thus x is in $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

(5) B) If $\sin\theta = \frac{2}{3}$ and $\frac{\pi}{2} < \theta < \pi$, give the exact value of $\cos\theta$.

Solution: We will use the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$. For $\frac{\pi}{2} < \theta < \pi$, $\cos\theta$ is negative. Thus $\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-\frac{4}{9}} = -\sqrt{\frac{5}{9}} = -\frac{\sqrt{5}}{3}$.

(5) C) Express as a single logarithm: $\frac{1}{2}\ln 3 - 3\ln 2 + \ln 6$.

Solution: $\frac{1}{2}\ln 3 - 3\ln 2 + \ln 6 = \ln 3^{1/2} - \ln 2^3 + \ln 6 = \ln\left(\frac{\sqrt{3}}{8}\cdot 6\right) = \ln\frac{3\sqrt{3}}{4}$.

(5) E) Complete the following table with the values for the functions f, g and h given that f is even, g is odd, and $h = f \circ g$.

Solution:

x	$f(x)$	$g(x)$	$h(x)$
-3	-1	0	1
-2	3	1	-2
-1	-2	-3	-1
0	1	2	3
1	-2	3	-1
2	3	-1	-2
3	-1	0	1

V. Consider the function $f(x) = 2e^{x+3} + 1$.

(5) A) What are the domain and range of f?

Solution: The domain of f is the set of all real numbers.

To determine the range of f we use the fact that $e^{x+3} > 0$. Thus $2e^{x+3} > 0$ and $2e^{x+3} + 1 > 1$. Thus the range of f is $(1, \infty)$.

(5) B) Given that f is one-to-one find a formula for the inverse function of f.

Solution: Start with $y = 2e^{x+3}+1$ and solve for x in terms of y. Thus $\dfrac{y-1}{2} = e^{x+3}$ and thus $x+3 = \ln\left(\dfrac{y-1}{2}\right)$. Therefore $x = \ln\left(\dfrac{y-1}{2}\right) - 3$ and thus $f^{-1}(x) = \ln\left(\frac{x-1}{2}\right) - 3$.

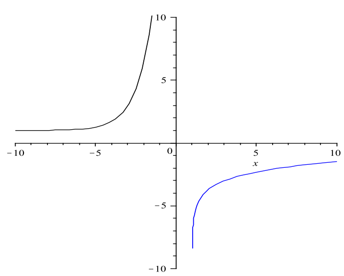
(5) C) What are the domain and range of f^{-1}?

Solution: The domain of f^{-1} is the range of f. So, the domain of f^{-1} is $(1, \infty)$.

The range of f^{-1} is the domain of f. So, the range of f^{-1} is the set of all real numbers.

(5) D) On the next page, plot the graph of the functions f and f^{-1} on the same set of axes. Label one point on each graph with its coordinates.

Solution:



The graph appearing on the left is the graph of f. It has a horizontal asymptote: $y = 1$. The point $(-3, 3)$ is on the graph. The graph will continue to the right. The y-intercept is $(0, 2e^3 + 1)$ which is approximately $(0, 41.17)$. The graph on the right is the graph of f^{-1}. It has a vertical asymptote: $x = 1$. The point $(3, -3)$ is on the graph. The x-intercept is $(2e^3 + 1, 0)$, or approximately $(41.71, 0)$

VI. The half-life of radioactive polonium-210 is about 140 days.

(5) A) If a sample contains 200g of polonium-210 originally, find the amount remaining after 560 days.

Solution: Let $f(t)$ be the amount of polonium-210 left after t days. Then $f(0) = 200$, $f(140) = \frac{1}{2} \cdot 200 = 100$, $f(280) = \frac{1}{2^2} \cdot 200 = 50$, $f(420) = \frac{1}{2^3} \cdot 200 = 25$, $f(560) = \frac{1}{2^4} \cdot 200 = 12.5$. After 560 days there will be about 12.5g of polonium-210 left.

(5) B) Find the amount remaining after t days.

Solution: Following the pattern from part (A), we see that $f(t) = \frac{1}{2^{t/140}} \cdot 200$

(5) C) How long will it take the amount in the sample to decay to 10g? (Give your answer in exact form.)

Solution: To answer this question we need to solve the equation

$$\frac{1}{2^{t/140}} \cdot 200 = 10.$$

Thus $2^{t/140} = 20$. Applying the *ln* function to both sides we obtain $\ln\left(2^{t/140}\right) = \ln 20$. Then $\frac{t}{140} \ln 2 = \ln 20$ and therefore it takes $t = \frac{140 \ln 20}{\ln 2}$ days until the sample will decay to 10g.