Mathematics 132 – Calculus for Physical and Life Sciences 2
Makeup Quiz Solutions – February 29, 2008

Some possibly useful formulas:

$$\sin^2\theta + \cos^2\theta = 1, \qquad \tan^2\theta + 1 = \sec^2\theta, \qquad \sin(2\theta) = 2\sin\theta\cos\theta.$$

(5) A) Integrate:

$$\int \frac{x^3}{\sqrt{9-x^2}}\,dx$$

Because of the $9-x^2$ under the radical sign, we use the sin substitution, letting $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$ and we proceed as follows:

$$\begin{aligned}
\int \frac{x^3}{\sqrt{9-x^2}}\,dx &= \int \frac{(3\sin\theta)^3}{\sqrt{9-(3\sin\theta)^2}}\,3\cos\theta\,d\theta \\
&= \int \frac{27\sin^3\theta \cdot 3\cos\theta}{\sqrt{9(1-\sin^2\theta)}}\,d\theta \\
&= \int \frac{27\sin^3\theta \cdot 3\cos\theta}{\sqrt{9\cos^2\theta}}\,d\theta \quad \text{(by the basic identity)} \\
&= \int \frac{27\sin^3\theta \cdot 3\cos\theta}{3\cos\theta}\,d\theta \\
&= 27\int \sin^3\theta\,d\theta \quad \text{(after cancelling)} \\
&= 27\int \sin^2\theta \cdot \sin\theta\,d\theta \quad \text{(odd power strategy)} \\
&= 27\int (1-\cos^2\theta)\sin\theta\,d\theta \\
&= 27\left(\int \sin\theta\,d\theta - \int \cos^2\theta\sin\theta\,d\theta\right) \quad \text{(multiply out and split)} \\
&= 27\left(-\cos\theta + \frac{1}{3}\cos^3\theta\right) + C
\end{aligned}$$

The last line comes from applying the u-substitution $u = \cos\theta$ to the second integral (note that $du = -\sin\theta\,d\theta$). Now we convert back to functions of x from the substitution $x = 3\sin\theta$. The reference triangle has opposite side x, hypotenuse 3 and adjacent side $\sqrt{9-x^2}$. Hence $\cos\theta = \sqrt{9-x^2}/3$, and the integral equals

$$= -9\sqrt{9-x^2} + \frac{1}{3}(9-x^2)^{3/2} + C.$$

(5) B) Integrate:

$$\int \frac{x^4+2x+1}{x^3+x}\,dx$$

Since the degree of the top is larger, we have to divide the denominator into the numerator. The quotient is x and the remainder is $x^4+2x+1-x(x^3+x) = -x^2+2x+1$. Therefore,

$$\frac{x^4+2x+1}{x^3+x} = x + \frac{-x^2+2x+1}{x^3+x} = x + \frac{-x^2+2x+1}{x(x^2+1)}.$$

We split the second term up into partial fractions like this:

$$\begin{aligned}
\frac{-x^2+2x+1}{x(x^2+1)} &= \frac{A}{x} + \frac{Bx+C}{x^2+1}, \quad \text{(so after clearing denoms)}\\
-x^2+2x+1 &= A(x^2+1) + x(Bx+C)\\
-x^2+2x+1 &= (A+B)x^2 + Cx + A
\end{aligned}$$

By comparing coefficients $A=1$, $C=2$ and $A+B=-1$. Hence $B=-2$. This gives

$$\begin{aligned}
x + \frac{-x^2+2x+1}{x(x^2+1)} &= x + \frac{1}{x} + \frac{-2x+2}{x^2+1} \quad \text{(so)}\\
\int x + \frac{-x^2+2x+1}{x(x^2+1)}\,dx &= \int x + \frac{1}{x} + \frac{-2x+2}{x^2+1}\,dx\\
&= \frac{x^2}{2} + \ln|x| + \int \frac{-2x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx\\
&= \frac{x^2}{2} + \ln|x| - \ln(x^2+1) + 2\tan^{-1}(x) + C
\end{aligned}$$

The third term here comes from the integral $\int \frac{-2x}{x^2+1}\,dx$, which is $-\int \frac{du}{u}$ for $u = x^2+1$. The last term is the basic inverse tangent integral.