MATH 131, section 1 – Calculus for Physical and Life Sciences
Solutions for Sample Exam Questions – Exam 3
November 26, 2007

I. Find y' and simplify.

(a) $y = \ln(x)\left(x^7 - \frac{4}{\sqrt{x}}\right)$

Solution: By the product rule,

$$\begin{aligned} y' &= \ln(x)(7x^6 + 2x^{-3/2}) + \frac{1}{x}\left(x^7 - \frac{4}{x^{1/2}}\right) \\ &= \ln(x)(7x^6 + 2x^{-3/2}) + x^6 - \frac{4}{x^{3/2}} \end{aligned}$$

(b) $y = (e^{2x} + 2)^3$.

Solution: By the chain rule (twice),

$$y' = 3(e^{2x} + 2)^2 \cdot e^{2x} \cdot 2 = 6e^{2x}(e^{2x} + 2)^2.$$

(c) $y = \frac{x+1}{3x^4-1}$.

Solution: By the quotient rule,

$$\begin{aligned} y' &= \frac{(3x^4 - 1)(1) - (x + 1)(12x^3)}{(3x^4 - 1)^2} \\ &= \frac{-9x^4 - 12x^3 - 1}{(3x^4 - 1)^2} \end{aligned}$$

(d) $y = \frac{\sin(x)}{1+\cos(x)}$

Solution: By the quotient rule,

$$\begin{aligned} y' &= \frac{(1 + \cos(x))(\cos(x)) - \sin(x)(-\sin(x))}{(1 + \cos(x))^2} \\ &= \frac{\cos(x) + \cos^2(x) + \sin^2(x)}{(1 + \cos(x))^2} \\ &= \frac{1}{1 + \cos(x)} \end{aligned}$$

(e) $y = \tan^{-1}(e^{5x})$.

Solution: By the derivative rule for inverse tangent and the chain rule,

$$y' = \frac{1}{1 + (e^{5x})^2} \cdot e^{5x} \cdot 5 = \frac{5e^{5x}}{1 + e^{10x}}.$$

(f) $xy^2 - 3y^3 + 2x^4 = 2$.

Solution: Since the equation involves both x and y we use *implicit differentiation.* Differentiating thinking of y as a function of x,

$$2xyy' + y^2 - 9y^2y' + 8x^3 = 0$$

Then solving for y':

$$y' = \frac{-y^2 - 8x^3}{2xy - 9y^2}.$$

(g) $y = \cos(x)^{x^3}$.

Solution: For functions of the form $u(x)^{v(x)}$, we use *logarithmic differentiation.* First take ln of both sides

$$\ln(y) = \ln(\cos(x)^{x^3}) = x^3 \ln(\cos(x)),$$

then differentiate using the product and chain rules:

$$\frac{1}{y}y' = -x^3 \tan(x) + 3x^2 \ln(\cos(x)).$$

So

$$y' = y(-x^3 \tan(x) + 3x^2 \ln(\cos(x))) = \cos(x)^{x^3}(-x^3 \tan(x) + 3x^2 \ln(\cos(x))).$$

II. The quantity of a reagent present in a chemical reaction is given by $Q(t) = t^3 - 3t^2 + t + 30$ grams at time t seconds for all $t \geq 0$.

(a) Over which intervals with $t \geq 0$ is the amount increasing? decreasing?

Solution: We need to determine the intervals where $Q'(t) = 3t^2 - 6t + 1$ is positive and negative. By the quadratic formula, $3t^2 - 6t + 1 = 0$ when

$$t = \frac{6 \pm \sqrt{24}}{6} = \frac{3 \pm \sqrt{6}}{3},$$

which are both positive numbers. By making a sign chart for Q' we see $Q'(t) > 0$ for t in $\left[0, \frac{3-\sqrt{6}}{3}\right) \cup \left(\frac{3+\sqrt{6}}{3}, +\infty\right)$. (Note the problem just said look at $t \geq 0$.) $Q'(t) < 0$ for t in $\left(\frac{3-\sqrt{6}}{3}, \frac{3+\sqrt{6}}{3}\right)$.

(b) Over which intervals is the rate of change of Q increasing? decreasing? The rate of change is increasing (decreasing) when $Q''(t) = 6t - 6$ is positive (negative). This is increasing for $t > 1$ and decreasing for $0 \leq t < 1$ (again, the problem said look only at $t \geq 0$ so we are ignoring $t < 0$).

III. A spherical balloon is being inflated at 20 cubic inches per minute. When the radius is 6 inches, at what rate is the radius of the balloon increasing? At what rate is the surface area increasing? (The volume of a sphere of radius r is $V = \frac{4\pi r^3}{3}$ and the surface area is $4\pi r^2$.)

Solution: This is a *related rates* problem. From the volume formula, since V and r are changing with time t,

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

We are given that $\frac{dV}{dt} = 20$, so when $r = 6$:

$$20 = 4\pi(6)^2 \frac{dr}{dt},$$

so

$$\frac{dr}{dt} = \frac{20}{144\pi} = \frac{5}{36\pi}$$

(inches per minute). The surface area is changing at the rate

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} = 8\pi \cdot 6 \cdot \frac{5}{36\pi} = \frac{20}{3}$$

(square inches per minute).

IV. All parts of this question refer to $f(x) = 4x^3 - x^4$.

(a) Find and classify all the critical numbers of f using the First Derivative Test.

Solution: The first derivative is $f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x)$, so f has critical numbers at $x = 0$ and $x = 3$. The first derivative is positive for $x < 0$, positive for $0 < x < 3$, and negative for $x > 3$. Hence $x = 3$ is a local maximum, and $x = 0$ is neither a local max nor a local min.

(b) Over which intervals is the graph $y = f(x)$ concave up? concave down?

Solution: For concavity, we need $f''(x) = 24x - 12x^2 = 12x(2 - x)$. This is negative for $x < 0$ and $x > 2$, and positive for $0 < x < 2$. Hence $y = f(x)$ is concave up on $(0, 2)$ and concave down on $(-\infty, 0) \cup (2, +\infty)$. (Since the graph changes concavity at $x = 0, 2$, these are *points of inflection.*)

(c) Sketch the graph $y = f(x)$.

Solution: Omitted – we will do this in the review session in class.

(d) Find the absolute maximum and minimum of $f(x)$ on the interval $[1, 4]$.

Solution: On this closed interval, we have a critical number at $x = 3$. The critical value is $f(3) = 4 \cdot 27 - 81 = 27$. The values at the endpoints are $f(1) = 4 - 1 = 3$ and $f(4) = 4 \cdot 64 - 256 = 0$. Hence the maximum value on the interval is $f(3) = 27$ and the minimum is $f(4) = 0$.

V. All three parts of this question refer to the function $f(x) = x^{2/3} - \frac{1}{5}x^{5/3}$.

(a) Find all the critical numbers of $f(x)$.

Solution: The derivative is

$$f'(x) = \frac{2}{3}x^{-1/3} - \frac{1}{3}x^{2/3} = \frac{1}{3}x^{-1/3}(2 - x)$$

The function has critical numbers at $x = 0$ ($f'(0)$ is undefined – the graph has a cusp point there), and $x = 2$ ($f'(2) = 0$).

(b) Find all the inflection points of $f(x)$.

Solution: For this we need

$$f''(x) = \frac{-2}{9}x^{-4/3} - \frac{2}{9}x^{-1/3} = -\frac{2}{9}x^{-4/3}(1 + x)$$

There is just one point of inflection at $x = -1$. Since $x^{-4/3} = \frac{1}{(x^{1/3})^4}$, that term is positive whenever it is defined, so the sign of f'' changes only at $x = -1$.

(c) For which of the critical numbers here is the Second Derivative Test applicable? Why? Determine the type of each such critical number using the Second Derivative Test.

Solution: The Second Derivative Test is applicable at critical numbers where the first derivative is equal to zero; here only at $x = 2$. We have $f''(2) = -\frac{2}{9} \cdot 2^{-4/3}(1 + 2) = -\frac{2}{3} \cdot 2^{-4/3} < 0$. Hence f has a *local maximum* at $x = 2$ since the graph is concave down there.

VI.

(a) A rectangular pen is to be constructed with one side along an existing long stone wall (so no fence is needed along that side), and the other three sides made of new fencing. The pen must enclose 1000 square feet of space. What is the smallest total length of fencing that can be used to enclose the pen? What are the dimensions of the pen that will use the smallest total length of fencing.

Solution: Let x be the length of the side of the pen parallel to the existing wall, and let y be the length of the other side. We have the area of the pen is $A = xy = 1000$, which implies $y = \frac{1000}{x}$. We want to minimize the total fencing used, which is

$$L = x + 2y = x + \frac{2000}{x}.$$

To minimize, we take the derivative of L and set it equal to zero to find the critical numbers:

$$L' = 1 - \frac{2000}{x^2} = 0$$

when $x = \sqrt{2000} \doteq 44.72$ feet. This is a minimum by the Second Derivative Test:

$$L'' = \frac{4000}{x^3} > 0$$

for all $x > 0$. (Note that $L \to +\infty$ as $x \to 0^+$ and as $x \to \infty$.) The smallest total length of fencing is

$$L(44.72) = 44.72 + \frac{2000}{44.72} \doteq 89.44$$

feet. The dimensions of the pen using the smallest amount of fencing are $x = 44.72$ and $y = \frac{1000}{44.72} \doteq 22.36$ feet.

(b) Consider all rectangles with one side along the x-axis that are inscribed under the curve $y = \sqrt{x}$, extending to $x = 4$ on the right. Find the dimensions of the rectangle of this type of largest area.

Solution: If the top left corner of the rectangle is at $(x, \sqrt{x})$, then the area is $A(x) = \sqrt{x}(4 - x) = 4x^{1/2} - x^{3/2}$. To find the maximum area, we take the derivative and set it equal to zero: $A'(x) = 2x^{-1/2} - (3/2)x^{1/2} = 0$. This has just one solution: $x = 4/3$. This gives a local maximum since $A''(x) = -x^{-3/2} - \frac{3}{4}x^{-1/2}$ is negative for all $x > 0$. The dimensions of the rectangle are base $= 4 - 4/3 = 8/3$ and height $= \sqrt{4/3} = 2/\sqrt{3}$.

VII. Evaluate the following limits.

(a)

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$$

Solution: This is indeterminate of the form $0/0$, so applying L'Hopital's Rule twice,

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \to 0} \frac{\sin(x)}{2x} = \lim_{x \to 0} \frac{\cos(x)}{2} = \frac{1}{2}.$$

(We could also use $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ to see this after applying L'Hopital's Rule the first time.)

(b)

$$\lim_{x \to \infty} \frac{\ln(1 + e^x)}{x}$$

Solution: This is indeterminate of the form ∞/∞, so applying L'Hopital's Rule,

$$\lim_{x \to \infty} \frac{\ln(1 + e^x)}{x} = \lim_{x \to \infty} \frac{\frac{e^x}{1+e^x}}{1} = \lim_{x \to \infty} \frac{1}{e^{-x} + 1} = 1.$$