1. Determine whether each of the following sequences converges or diverges. If it converges, find its limit.

(a) \[ a_n = \frac{1 - n^2}{2 + 3n^2} \]

\text{Solution:} \quad \lim_{n \to \infty} \frac{1 - n^2}{2 + 3n^2} = \lim_{n \to \infty} \frac{\frac{1}{n^2} - 1}{\frac{2}{n^2} + 3} = \frac{1}{3}.

Or use l’Hospital’s rule twice: \[ \lim_{n \to \infty} \frac{1 - n^2}{2 + 3n^2} = \lim_{n \to \infty} \frac{1 - x^2}{2 + 3x^2} = \lim_{n \to \infty} \frac{-2x}{6x} = \lim_{n \to \infty} \frac{-2}{6} = \frac{1}{3} \]

(b) \[ a_n = 1 + (-1)^n \]

\text{Solution:} \quad a_n = 0 \text{ if } n \text{ is odd and } a_n = 2 \text{ if } n \text{ is even}. The sequence does not approach a single number and thus it diverges.

(c) \[ a_n = \frac{\sin n}{3^n} \]

\text{Solution:} \quad \text{Use the Squeeze Theorem. } -1 \leq \sin n \leq 1 \text{ for all } n. \text{ Thus } \frac{-1}{3^n} \leq \frac{\sin n}{3^n} \leq \frac{1}{3^n}.

Since \[ \lim_{n \to \infty} \frac{-1}{3^n} = \lim_{n \to \infty} \frac{1}{3^n} = 0, \text{ we have that } \lim_{n \to \infty} \frac{\sin n}{3^n} = 0. \]

(d) \[ a_n = \frac{(\ln n)^2}{n} \]

\text{Solution:} \quad \lim_{n \to \infty} \frac{(\ln n)^2}{n} = \lim_{x \to \infty} \frac{(\ln x)^2}{x} = \lim_{x \to \infty} \frac{2(\ln x)}{x} = \lim_{x \to \infty} \frac{2 \ln x}{x} = \lim_{x \to \infty} \frac{2}{x} = 0.

(e) \[ a_n = \frac{2^n + 1}{e^n} \]

\text{Solution:} \quad \lim_{n \to \infty} \frac{2^n + 1}{e^n} = \lim_{n \to \infty} \left( \frac{2}{e} \right)^n + \lim_{n \to \infty} \left( \frac{1}{e} \right)^n = 0 \text{ since both } \frac{2}{e} \text{ and } \frac{1}{e} \text{ are between 0 and 1.}

(f) \[ a_n = \ln(2n + 1) - \ln(2n - 1) \]

\text{Solution:} \quad \lim_{n \to \infty} a_n = \lim_{n \to \infty} \ln \left( \frac{2n + 1}{2n - 1} \right) = \ln \left( \lim_{n \to \infty} \frac{2n + 1}{2n - 1} \right) = 0.
2. Determine if each infinite series converges or diverges. If it converges, find its sum. (Be careful about the first term of the series.)

(a) \(1 + e^{-1} + e^{-2} + e^{-3} + \ldots + e^{-n} + \ldots\)

\textit{Solution:} \(\sum_{n=0}^{\infty} (e^{-1})^n\) is the geometric series with ratio \(\frac{1}{e}\) and first term 1. It is convergent and its sum is \(\frac{e}{e - 1}\).

(b) \(\sum_{n=0}^{\infty} \frac{3^n - 2^n}{4^n}\)

\textit{Solution:} \(\sum_{n=0}^{\infty} \frac{3^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = 4\) (geometric series with ratio 3/4 and first term 1). \(\sum_{n=0}^{\infty} \left(\frac{2}{4}\right)^n = 2\) (geometric series with ratio 1/2 and first term 1).

Thus \(\sum_{n=0}^{\infty} \frac{3^n - 2^n}{4^n} = \sum_{n=0}^{\infty} \frac{3^n}{4^n} - \sum_{n=0}^{\infty} \frac{2^n}{4^n} = 4 - 2 = 2\)

(c) \(\sum_{n=0}^{\infty} \left(\frac{100}{99}\right)^n\)

\textit{Solution:} This is a geometric series with ratio 100/99 > 1. It diverges.

(d) \(\sum_{n=0}^{\infty} \left(\frac{99}{100}\right)^n\)

\textit{Solution:} This is a geometric series with ratio 99/100 < 1 and first term 1. It converges to 100.

(e) \(\sum_{n=1}^{\infty} \left(\frac{\pi}{e}\right)^n\)

\textit{Solution:} This is a geometric series with ratio \(\pi/e > 1\). It diverges.

(f) \(\sum_{n=1}^{\infty} \left(\frac{2}{n} - \frac{1}{2^n}\right)\)

\textit{Solution:} \(\sum_{n=0}^{\infty} \frac{2}{n} = 2 \sum_{n=0}^{\infty} \frac{1}{n}\) is divergent (twice the harmonic series). \(\sum_{n=0}^{\infty} \frac{1}{2^n}\) converges (geometric series with ratio 1/2). Thus \(\sum_{n=1}^{\infty} \left(\frac{2}{n} - \frac{1}{2^n}\right)\) diverges. (If \(\sum_{n=1}^{\infty} \left(\frac{2}{n} - \frac{1}{2^n}\right)\) were convergent,
\[
\sum_{n=0}^{\infty} \frac{2}{n} = \sum_{n=1}^{\infty} \left( \frac{2}{n} - \frac{1}{2n} \right) + \sum_{n=0}^{\infty} \frac{1}{2^n}
\]
would be convergent as the sum of two convergent series.

\[(g) \sum_{k=1}^{\infty} \left( \frac{k + 1}{2k + 4} \right)\]

**Solution:** Since \(\lim_{k \to \infty} \frac{k + 1}{2k + 4} = \frac{1}{2} \neq 0\), the series diverges by the Divergence Test.

\[(h) \sum_{n=1}^{\infty} \ln \left( \frac{n + 1}{n} \right)\]

(Hint: Consider several partial sums and use properties of logarithms to simplify them.)

**Solution:** Use the fact that \(\ln \left( \frac{n + 1}{n} \right) = \ln(n + 1) - \ln n\). Consider the \(n\)-th partial sum
\[s_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + \cdots + (\ln n - \ln(n - 1)) + (\ln(n + 1) - \ln n)\]
All terms cancel except for \(\ln 1 + \ln(n + 1)\). Thus \(s_n = \ln(n + 1) - \ln 1 = \ln(n + 1)\). Since \(\lim_{n \to \infty} s_n = \infty\), the series diverges.

3. Find the rational number represented by 0.2525252525…
(Hint: Think of 0.2525252525… as a geometric series.)

**Solution:** 0.2525252525… = \(\frac{25}{100} + \frac{25}{(100)^2} + \frac{25}{(100)^3} + \cdots\) This is the geometric series with ratio \(\frac{1}{100}\) and first term \(25/100\). It is convergent and its sum is
\[\frac{25}{1 - \frac{1}{100}} = \frac{25}{99}\]
Thus 0.2525252525… = \(\frac{25}{99}\).

4. Suppose the government spends $1 billion and that each recipient spends 90% of the dollars that he or she receives. In turn, the secondary recipients spend 90% of the dollars they receive, and so on. How much total spending results from the original injection of $1 billion into the country?

**Solution:** The total spending (in Dollars) equals \(10^9 + 10^9 \cdot 0.9 + 10^9(0.9)^2 + 10^9(0.9)^3 + \cdots = \frac{10^9}{1 - 0.9} = 9 \cdot 10^9\) since this is a geometric series with ratio 0.9 and first term \(10^9\). Thus the total spending is $10 billion.

5. Use the integral test to decide whether each of the following series converges or diverges.
(a) \( \sum_{n=0}^{\infty} \frac{n}{n^2 + 1} \)

**Solution:** Let \( f(x) = \frac{x}{x^2 + 1} \). By the quotient rule (check!) \( f'(x) = \frac{1 - x^2}{(x^2 + 1)^2} \). For \( x > 1 \) this is negative and thus \( f \) is decreasing. The function \( f \) is also continuous and positive for \( x > 0 \). We can apply the integral test (use the \( u \)-substitution \( u = x^2 + 1, \ du = 2x \, dx \) in the integral).

\[
\int_{1}^{\infty} \frac{x}{x^2 + 1} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{x}{x^2 + 1} \, dx = \lim_{b \to \infty} \frac{1}{2} \int_{2}^{b^2 + 1} \frac{1}{u} \, du = \lim_{b \to \infty} \frac{1}{2} (\ln(b^2 + 1) - \ln 2) = \infty.
\]

Since the integral is divergent, the series is divergent as well.

(b) \( \sum_{n=0}^{\infty} \frac{n}{e^{n^2}} \)

**Solution:** Let \( f(x) = \frac{x}{e^x} \). We have \( f'(x) = \frac{e^{x^2} - 2xe^{x^2}}{(e^{x^2})^2} = \frac{e^{x^2} (1 - 2x^2)}{e^{2x^2}} \). Since \( f'(x) < 0 \) for \( x \geq 1 \), the function \( f \) is decreasing. The function \( f \) is also continuous and positive for \( x > 0 \) and we can apply the integral test (use the \( u \)-substitution \( u = x^2, \ du = 2x \, dx \) in the integral).

\[
\int_{1}^{\infty} \frac{x}{e^{x^2}} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{x}{e^{x^2}} \, dx = \lim_{b \to \infty} \frac{1}{2} \int_{1}^{b^2} e^{-u} \, du = \lim_{b \to \infty} \frac{1}{2} (-e^{-b^2} + e^{-1}) = \frac{1}{2} e^{-1} = \frac{1}{2e}
\]

Since the integral is convergent, the series converges as well.

(c) \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \)

**Solution:** Let \( f(x) = \frac{1}{x \ln x} \). Since the denominator is increasing, the function is decreasing. It is positive and continuous for \( x > 1 \) and we can apply the integral test (use the \( u \)-substitution \( u = \ln x, \ du = 1/x \, dx \) in the integral).

\[
\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} \int_{\ln 2}^{\ln b} \frac{1}{u} \, du = \lim_{b \to \infty} (\ln(\ln b) - \ln(\ln 2)) = \infty.
\]

Since the integral diverges, the series diverges as well.

(d) \( \sum_{n=1}^{\infty} \frac{n}{e^n} \)

**Solution:** Let \( f(x) = \frac{x}{e^x} \). Since \( f'(x) = \frac{e^x (1-x)}{e^{2x}} \) is negative for \( x > 1 \), the function \( f \) is
decreasing for $x > 1$. Also, for $x > 0$, $f$ is continuous and positive. We can apply the integral test (use integration by parts in the definite integral: $u = x\, du = 1\, dx$ and $dv = e^{-x}\, dx$, $v = -e^{-x}$).

$$\int_1^\infty \frac{x}{e^x}\, dx = \lim_{b \to \infty} \int_1^b x e^{-x}\, dx = \lim_{b \to \infty} (-xe^{-x}|_1^b + \int_1^b e^{-x}\, dx) = \lim_{b \to \infty} (-be^{-b} - e^{-b} + 2e^{-1}).$$

By l'Hospital’s rule $\lim_{b \to \infty} be^{-b} = \lim_{b \to \infty} \frac{b}{e^b} = \lim_{b \to \infty} \frac{1}{e^b} = 0$. Thus $\int_1^\infty \frac{x}{e^x}\, dx = 2e^{-1}$. Since the integral is convergent, the series is convergent as well.

$$(e) \sum_{n=1}^\infty \ln \left(1 + \frac{1}{n^2}\right)$$

(Hint: integrate by parts.)

**Solution:** Let $f(x) = \ln \left(1 + \frac{1}{x^2}\right)$. It is a positive, continuous and decreasing function for $x \geq 1$ and we can apply the integral test. We will first solve $\int \ln \left(1 + \frac{1}{x^2}\right)\, dx$ using integration by parts. Let $u = \ln \left(1 + \frac{1}{x^2}\right)$ and $dv = dx$. Then $du = \frac{1}{1 + \frac{1}{x^2}} \cdot \frac{-2}{x^3}\, dx = -\frac{2}{x(x^2 + 1)}\, dx$ and $v = x$. Then $\int \ln \left(1 + \frac{1}{x^2}\right)\, dx = x\ln \left(1 + \frac{1}{x^2}\right) + 2 \int \frac{1}{1 + x^2}\, dx = x\ln \left(1 + \frac{1}{x^2}\right) + 2\arctan x + C$.

Then $\int_1^\infty \ln \left(1 + \frac{1}{x^2}\right)\, dx = \lim_{b \to \infty} \int_1^b \ln \left(1 + \frac{1}{x^2}\right)\, dx = \lim_{b \to \infty} \left(\int_1^b \ln \left(1 + \frac{1}{x^2}\right)\, dx\right) + 2\arctan b - \ln 2 + 2\arctan 2 = \\
\lim_{b \to \infty} b\ln \left(1 + \frac{1}{b^2}\right) = \lim_{b \to \infty} \ln \left(1 + \frac{1}{b^2}\right) = \lim_{b \to \infty} \frac{1}{1 + \frac{1}{b^2}} = \lim_{b \to \infty} \frac{2}{b(b + 1/b^2)} = 0$

We have $\lim_{b \to \infty} b\ln \left(1 + \frac{1}{b^2}\right) = \lim_{b \to \infty} \ln \left(1 + \frac{1}{b^2}\right) = \lim_{b \to \infty} \frac{1}{1 + \frac{1}{b^2}} = \lim_{b \to \infty} \frac{2}{b(1 + 1/b^2)} = 0$ and $\lim_{b \to \infty} \arctan b = \pi/2$.

Therefore the integral converges and so does the series.

6. Determine whether the given series converges or diverges. Quote general results to justify your answers.

(a) $\sum_{n=1}^\infty \frac{1}{(\sqrt{n})^5}$

**Solution:** $\sum_{n=1}^\infty \frac{1}{(\sqrt{n})^5} = \sum_{n=1}^\infty \frac{1}{n^{5/2}}$. This is the $p$-series with $p = 5/2 > 1$ and is therefore convergent.

(b) $\sum_{n=1}^\infty \frac{n + 1}{n^2}$
**Solution:** Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges (harmonic series) and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) convergent \((p\text{-series with } p = 2)\), the given series diverges.

7. Show that the series \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \) converges if \( p > 1 \) and diverges if \( p \leq 1 \).

**Solution:** If \( p = 1 \) this is the series of problem 5.c) which diverges. Suppose \( p \neq 1 \). Since \( f(x) = \frac{1}{x(\ln x)^p} \) is positive, continuous and decreasing for \( x > 1 \) we use the integral test. In the integral we use the substitution \( u = \ln x, \ du = 1/x \, dx \).

\[
\int_{2}^{\infty} \frac{1}{x(\ln x)^p} \, dx = \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x(\ln x)^p} \, dx = \lim_{b \to \infty} \int_{\ln 2}^{\ln b} \frac{1}{u^p} \, du = \lim_{b \to \infty} \frac{u^{-p+1}}{-p+1} \bigg|_{\ln 2}^{\ln b} = \\
= \lim_{b \to \infty} \left( \frac{(\ln b)^{-p+1}}{-p+1} - \frac{(\ln 2)^{-p+1}}{-p+1} \right)
\]

If \( p > 1 \), then \( \lim_{b \to \infty} (\ln b)^{-p+1} = 0 \) and if \( p < 1 \), then \( \lim_{b \to \infty} (\ln b)^{-p+1} = \infty \). Therefore, the integral converges is \( p > 1 \) and it diverges if \( p < 1 \). By the integral test and problem 5.c), the series converges if \( p > 1 \) and it diverges if \( p \leq 1 \).

8. For each of the following series explain why the integral test does not apply.

(a) \( \sum_{n=1}^{\infty} e^{-n} \sin n \)

**Solution:** The integral test does not apply because the function \( f(x) = e^{-x} \sin x \) is not positive (for example it takes negative values at each odd multiple of \( \pi/2 \)) and not decreasing \((f'(x) = e^{-x}(\cos x - \sin x) \) is not always negative - for example, it is positive at all even multiples of \( \pi \)).

(b) \( \sum_{n=1}^{\infty} \frac{2 + \sin n}{n^2} \)

**Solution:** Here the function \( f(x) = \frac{2 + \sin x}{x^2} \) is positive for all \( x \neq 0 \) (because \( -1 \leq \sin x \leq 1 \)). However, the function is not decreasing since \( f'(x) = \frac{x^2 \cos x - 2x(2 + \sin x)}{x^4} \) is not always negative (it is positive for all even multiples of \( \pi \) greater than 4).