Math 132: Calculus for Physical and Life Sciences 2
Problem Set 8
Due Friday, April 11, 2008, at the beginning of class.
Solutions

1. Determine whether each of the following sequences converges or diverges. If it converges, find its limit.

(a) $a_n = \frac{1-n^2}{2+3n^2}$

Solution: $\lim_{n\to\infty} \frac{1-n^2}{2+3n^2} = \lim_{n\to\infty} \frac{\frac{1}{n^2}-1}{\frac{2}{n^2}+3} = -\frac{1}{3}$.

Or use l'Hospital's rule twice: $\lim_{n\to\infty} \frac{1-n^2}{2+3n^2} = \lim_{x\to\infty} \frac{1-x^2}{2+3x^2} \overset{l'H}{=} \lim_{x\to\infty} \frac{-2x}{6x} \overset{l'H}{=} \lim_{x\to\infty} \frac{-2}{6} = -\frac{1}{3}$

(b) $a_n = 1 + (-1)^n$

Solution: $a_n = 0$ if n is odd and $a_n = 2$ if n is even. The sequence does not approach a single number and thus it diverges.

(c) $a_n = \frac{\sin n}{3^n}$

Solution: Use the Squeeze Theorem. $-1 \le \sin n \le 1$ for all n. Thus $-\frac{1}{3^n} \le \frac{\sin n}{3^n} \le \frac{1}{3^n}$. Since $\lim_{n\to\infty} -\frac{1}{3^n} = \lim_{n\to\infty} \frac{1}{3^n} = 0$, we have that $\lim_{n\to\infty} \frac{\sin n}{3^n} = 0$.

(d) $a_n = \frac{(\ln n)^2}{n}$

Solution: $\lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{x\to\infty} \frac{(\ln x)^2}{x} \overset{l'H}{=} \lim_{x\to\infty} \frac{2(\ln x)\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln x}{x} \overset{l'H}{=} \lim_{x\to\infty} \frac{2}{x} = 0$.

(e) $a_n = \frac{2^n+1}{e^n}$

Solution: $\lim_{n\to\infty} \frac{2^n+1}{e^n} = \lim_{n\to\infty} \left(\frac{2}{e}\right)^n + \lim_{n\to\infty} \left(\frac{1}{e}\right)^n = 0$ since both $\frac{2}{e}$ and $\frac{1}{e}$ are between 0 and 1.

(f) $a_n = \ln(2n+1) - \ln(2n-1)$

Solution: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{2n+1}{2n-1}\right) = \ln\left(\lim_{n\to\infty} \frac{2n+1}{2n-1}\right) = 0$.

2. Determine if each infinite series converges or diverges. If it converges, find its sum. (Be careful about the first term of the series.)

(a) $1 + e^{-1} + e^{-2} + e^{-3} + \ldots + e^{-n} + \ldots$

Solution: $\sum_{n=0}^{\infty} (e^{-1})^n$ is the geometric series with ratio $\frac{1}{e}$ and first term 1. It is convergent and its sum is $\frac{e}{e-1}$.

(b) $\sum_{n=0}^{\infty} \frac{3^n - 2^n}{4^n}$

Solution: $\sum_{n=0}^{\infty} \frac{3^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = 4$ (geometric series with ratio 3/4 and first term 1). $\sum_{n=0}^{\infty} \left(\frac{2}{4}\right)^n =$ 2 (geometric series with ratio 1/2 and first term 1).

Thus $\sum_{n=0}^{\infty} \frac{3^n - 2^n}{4^n} = \sum_{n=0}^{\infty} \frac{3^n}{4^n} - \sum_{n=0}^{\infty} \frac{2^n}{4^n} = 4 - 2 = 2$

(c) $\sum_{n=0}^{\infty} \left(\frac{100}{99}\right)^n$

Solution: This is a geometric series with ratio $100/99 > 1$. It diverges.

(d) $\sum_{n=0}^{\infty} \left(\frac{99}{100}\right)^n$

Solution: This is a geometric series with ratio $99/100 < 1$ and first term 1. It converges to 100.

(e) $\sum_{n=1}^{\infty} \left(\frac{\pi}{e}\right)^n$

Solution: This is a geometric series with ratio $\pi/e > 1$. It diverges.

(f) $\sum_{n=1}^{\infty} \left(\frac{2}{n} - \frac{1}{2^n}\right)$

Solution: $\sum_{n=0}^{\infty} \frac{2}{n} = 2\sum_{n=0}^{\infty} \frac{1}{n}$ is divergent (twice the harmonic series). $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges (geometric series with ratio 1/2). Thus $\sum_{n=1}^{\infty} \left(\frac{2}{n} - \frac{1}{2^n}\right)$ diverges. (If $\sum_{n=1}^{\infty} \left(\frac{2}{n} - \frac{1}{2^n}\right)$ were convergent,

$\sum_{n=0}^{\infty} \frac{2}{n} = \sum_{n=1}^{\infty} \left(\frac{2}{n} - \frac{1}{2^n} \right) + \sum_{n=0}^{\infty} \frac{1}{2^n}$ would be convergent as the sum of two convergent series.)

(g) $\sum_{k=1}^{\infty} \left(\frac{k+1}{2k+4} \right)$

Solution: Since $\lim_{k \to \infty} \frac{k+1}{2k+4} = \frac{1}{2} \neq 0$, the series diverges by the Divergence Test.

(h) $\sum_{n=1}^{\infty} \ln \left(\frac{n+1}{n} \right)$

(Hint: Consider several partial sums and use properties of logarithms to simplify them.)

Solution: Use the fact that $\ln \left(\frac{n+1}{n} \right) = \ln(n+1) - \ln n$. Consider the n-th partial sum $s_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + \cdots + (\ln n - \ln(n-1)) + (\ln(n+1) - \ln n)$. All terms cancel except for $\ln 1$ and $\ln(n+1)$. Thus $s_n = \ln(n+1) - \ln 1 = \ln(n+1)$. Since $\lim_{n \to \infty} s_n = \infty$, the series diverges.

3. Find the rational number represented by 0.2525252525....

   (Hint: Think of 0.2525252525... as a geometric series.)

   Solution: $0.2525252525\ldots = \frac{25}{100} + \frac{25}{(100)^2} + \frac{25}{(100)^3} + \cdots$ This is the geometric series with ratio 1/100 and first term 25/100. It is convergent and its sum is

$$\frac{\frac{25}{100}}{1 - \frac{1}{100}} = \frac{25}{99}.$$

   Thus $0.2525252525\ldots = \frac{25}{99}$.

4. Suppose the government spends \$1 billion and that each recipient spends 90% of the dollars that he or she receives. In turn, the secondary recipients spend 90% of the dollars they receive, and so on. How much total spending results from the original injection of \$1 billion into the country?

   Solution: The total spending (in Dollars) equals $10^9 + 10^9 \cdot 0.9 + 10^9(0.9)^2 + 10^9(0.9)^3 + \cdots = \frac{10^9}{1-0.9} = 9 \cdot 10^9$ since this is a geometric series with ratio 0.9 and first term 10^9. Thus the total spending is \$10 billion.

5. Use the integral test to decide whether each of the following series converges or diverges.

(a) $\sum_{n=0}^{\infty} \frac{n}{n^2+1}$

Solution: Let $f(x) = \frac{x}{x^2+1}$. By the quotient rule (check!) $f'(x) = \frac{1-x^2}{(x^2+1)^2}$. For $x > 1$ this is negative and thus f is decreasing. The function f is also continuous and positive for $x > 0$. We can apply the integral test (use the u-substitution $u = x^2+1,\ du = 2x\,dx$ in the integral).

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \frac{1}{2} \int_2^{b^2+1} \frac{1}{u}\,du = \lim_{b\to\infty} \frac{1}{2}(\ln(b^2+1) - \ln 2) = \infty.$$

Since the integral is divergent, the series is divergent as well.

(b) $\sum_{n=0}^{\infty} \frac{n}{e^{n^2}}$

Solution: Let $f(x) = \frac{x}{e^{x^2}}$. We have $f'(x) = \frac{e^{x^2} - 2x^2e^{x^2}}{(e^{x^2})^2} = \frac{e^{x^2}(1-2x^2)}{e^{2x^2}}$. Since $f'(x) < 0$ for $x \geq 1$, the function f is decreasing. The function f is also continuous and positive for $x > 0$ and we can apply the integral test (use the u-substitution $u = x^2,\ du = 2x\,dx$ in the integral).

$$\int_1^{\infty} \frac{x}{e^{x^2}}\,dx = \lim_{b\to\infty} \int_1^b \frac{x}{e^{x^2}}\,dx = \lim_{b\to\infty} \frac{1}{2} \int_1^{b^2} e^{-u}\,du = \lim_{b\to\infty} \frac{1}{2}(-e^{-b^2} + e^{-1}) = \frac{1}{2}e^{-1} = \frac{1}{2e}$$

Since the integral is convergent, the series converges as well.

(c) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Solution: Let $f(x) = \frac{1}{x \ln x}$. Since the denominator is increasing, the function is decreasing. It is positive and continuous for $x > 1$ and we can apply the integral test (use the u-substitution $u = \ln x,\ du = 1/x\,dx$ in the integral).

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u}\,du = \lim_{b\to\infty} (\ln(\ln b) - \ln(\ln 2)) = \infty.$$

Since the integral diverges, the series diverges as well.

(d) $\sum_{n=1}^{\infty} \frac{n}{e^n}$

Solution: Let $f(x) = \frac{x}{e^x}$. Since $f'(x) = \frac{e^x(1-x)}{e^{2x}}$ is negative for $x > 1$, the function f is decreasing for $x > 1$. Also, for $x > 0$, f is continuous and positive. We can apply the integral test (use integration by parts in the definite integral: $u = x\, du = 1\, dx$ and $dv = e^{-x}\, dx, v = -e^{-x}$).

$$\int_1^{\infty} \frac{x}{e^x}\, dx = \lim_{b\to\infty} \int_1^b x e^{-x}\, dx = \lim_{b\to\infty} (-xe^{-x}|_1^b + \int_1^b e^{-x}\, dx = \lim_{b\to\infty} (-be^{-b} - e^{-b} + 2e^{-1}).$$

By l'Hospital's rule $\lim_{b\to\infty} be^{-b} = \lim_{b\to\infty} \frac{b}{e^b} = \lim_{b\to\infty} \frac{1}{e^b} = 0$. Thus $\int_1^{\infty} \frac{x}{e^x}\, dx = 2e^{-1}$. Since the integral is convergent, the series is convergent as well.

(e) $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$

(Hint: integrate by parts.)

Solution: Let $f(x) = \ln\left(1 + \frac{1}{x^2}\right)$. It is a positive, continuous and decreasing function for $x \geq 1$ and we can apply the integral test. We will first solve $\int \ln\left(1 + \frac{1}{x^2}\right)\, dx$ using integration by parts. Let $u = \ln\left(1 + \frac{1}{x^2}\right)$ and $dv = dx$. Then $du = \frac{1}{1+\frac{1}{x^2}} \cdot \frac{-2}{x^3}\, dx = -\frac{2}{x(x^2+1)}\, dx$ and $v = x$. Then $\int \ln\left(1 + \frac{1}{x^2}\right)\, dx = x \ln\left(1 + \frac{1}{x^2}\right) + 2\int \frac{1}{1+x^2}\, dx = x \ln\left(1 + \frac{1}{x^2}\right) + 2 \arctan x + C$.
Then

$$\int_1^{\infty} \ln\left(1 + \frac{1}{x^2}\right)\, dx = \lim_{b\to\infty} \int_1^b \ln\left(1 + \frac{1}{x^2}\right)\, dx = \lim_{b\to\infty} \left(b \ln\left(1 + \frac{1}{b^2}\right) + 2 \arctan b - \ln 2 + 2 \arctan 2\right.$$

We have $\lim_{b\to\infty} b \ln\left(1 + \frac{1}{b^2}\right) = \lim_{b\to\infty} \frac{\ln\left(1 + \frac{1}{b^2}\right)}{\frac{1}{b}} \overset{l'H}{=} \lim_{b\to\infty} \frac{\frac{1}{1+\frac{1}{b^2}} \cdot \frac{-2}{b^3}}{-\frac{1}{b^2}} = \lim_{b\to\infty} \frac{2}{b(1+\frac{1}{b^2})} = 0$ and $\lim_{b\to\infty} \arctan b = \pi/2$.
Therefore the integral converges and so does the series.

6. Determine whether the given series converges or diverges. Quote general results to justify your answers.

(a) $\sum_{n=1}^{\infty} \frac{1}{(\sqrt{n})^5}$

Solution: $\sum_{n=1}^{\infty} \frac{1}{(\sqrt{n})^5} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$. This is the p-series with $p = 5/2 > 1$ and is therefore convergent.

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$

Solution: Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series) and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent (p-series with $p = 2$), the given series diverges.

7. Show that the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Solution: If $p = 1$ this is the series of problem 5.c) which diverges. Suppose $p \neq 1$. Since $f(x) = \frac{1}{x(\ln x)^p}$ is positive, continuous and decreasing for $x > 1$ we use the integral test. In the integral we use the substitution $u = \ln x$, $du = 1/x\, dx$.

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\, dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^p}\, dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u^p}\, du = \lim_{b\to\infty} \left.\frac{u^{-p+1}}{-p+1}\right|_{\ln 2}^{\ln b} =$$

$$= \lim_{b\to\infty} \left(\frac{(\ln b)^{-p+1}}{-p+1} - \frac{(\ln 2)^{-p+1}}{-p+1} \right)$$

If $p > 1$, then $\lim_{b\to\infty} (\ln b)^{-p+1} = 0$ and if $p < 1$, then $\lim_{b\to\infty} (\ln b)^{-p+1} = \infty$. Therefore, the integral converges is $p > 1$ and it diverges if $p < 1$. By the integral test and problem 5.c), the series converges if $p > 1$ and it diverges if $p \leq 1$.

8. For each of the following series explain why the integral test does *not* apply.

(a) $\sum_{n=1}^{\infty} e^{-n} \sin n$

Solution: The integral test does not apply because the function $f(x) = e^{-x} \sin x$ is not positive (for example it takes negative values at each odd multiple of $\pi/2$) and not decreasing ($f'(x) = e^{-x}(\cos x - \sin x)$ is not always negative - for example, it is positive at all even multiples of π).

b) $\sum_{n=1}^{\infty} \frac{2 + \sin n}{n^2}$

Solution: Here the function $f(x) = \frac{2 + \sin x}{x^2}$ is positive for all $x \neq 0$ (because $-1 \leq \sin x \leq 1$). However, the function is not decreasing since $f'(x) = \frac{x^2 \cos x - 2x(2 + \sin x)}{x^4}$ is not always negative (it is positive for all even multiples of π greater than 4).