Math 132: Calculus for Physical and Life Sciences 2
Problem Set 5
Due Friday, February 29, 2008, at the beginning of class.

General Directions: You must show all work for credit on these problems.

1. Explain why each of the following integrals is an improper integral and determine whether each one converges or diverges. Evaluate those that are convergent.

(a) $\displaystyle\int_4^{\infty} \frac{1}{x^{3/2}}\,dx$

Solution: Integral is improper because the interval of integration is infinite.

$$\lim_{b\to\infty}\int_4^b \frac{1}{x^{3/2}}\,dx = \lim_{b\to\infty} \left.\frac{x^{-1/2}}{-1/2}\right|_4^b = \lim_{b\to\infty}\left(-\frac{2}{\sqrt{b}}+1\right) = 1.$$

The integral converges to 1.

(b) $\displaystyle\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$

Solution: Integral is improper because the integrand has an infinite discontinuity at $x = 4$ (integrand has vertical asymptote at $x = 4$).

$$\lim_{b\to 4^-}\int_0^b \frac{1}{\sqrt{4-x}}\,dx = \lim_{b\to 4^-} \left.-2\sqrt{4-x}\right|_0^b = \lim_{b\to 4^-}(-2\sqrt{4-b}+4) = 4.$$

The integral converges to 4.

(c) $\displaystyle\int_0^2 \frac{x}{x^2-1}\,dx$

Solution: Integral is improper because the integrand has an infinite discontinuity at $x = 1$ (integrand has vertical asymptote at $x = 1$).

$$\int_0^2 \frac{x}{x^2-1}\,dx = \int_0^1 \frac{x}{x^2-1}\,dx + \int_1^2 \frac{x}{x^2-1}\,dx$$

To see if the first integral converges, consider

$$\lim_{b\to 1^-}\int_0^b \frac{x}{x^2-1}\,dx = \lim_{b\to 1^-} \left.\frac{1}{2}\ln|x^2-1|\right|_0^b = \lim_{b\to 1^-}(\frac{1}{2}\ln|b^2-1| - \frac{1}{2}\ln 1) = -\infty.$$

Thus the first integral is divergent. Then, $\displaystyle\int_0^2 \frac{x}{x^2-1}\,dx$ is divergent.

(d) $\displaystyle\int_e^\infty \frac{1}{x\ln x}\,dx$

Solution: Integral is improper because the interval of integration is infinite.

$$\lim_{b\to\infty}\int_e^b \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \ln|\ln x|\Big|_e^b = \lim_{b\to\infty}(\ln|\ln b| - \ln|\ln e|) = \infty.$$

The integral diverges.

(e) $\displaystyle\int_0^\infty xe^{-x^2}\,dx$

Solution: Integral is improper because the interval of integration is infinite.

$$\lim_{b\to\infty}\int_0^b xe^{-x^2}\,dx = \lim_{b\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^b = \lim_{b\to\infty} -\frac{1}{2}(e^{-b^2}+\frac{1}{2}) = \frac{1}{2}.$$

The integral converges to $\dfrac{1}{2}$.

(f) $\displaystyle\int_0^\infty \sin^2 x\,dx$

Solution: Integral is improper because the interval of integration is infinite.

$$\lim_{b\to\infty}\int_0^b \sin^2 x\,dx = \lim_{b\to\infty}\int_0^b \frac{1-\cos(2x)}{2}\,dx = \lim_{b\to\infty}\left(\frac{1}{2}x - \frac{1}{4}\sin(2x)\right)\Big|_0^b =$$

$\displaystyle\lim_{b\to\infty}\left(\frac{1}{2}b - \frac{1}{4}\sin(2b)\right)$. Since $-\dfrac{1}{4} \le \dfrac{1}{4}\sin(2b) \le \dfrac{1}{4}$ and $\displaystyle\lim_{b\to\infty}\frac{1}{2}b = \infty$, we have

$\displaystyle\lim_{b\to\infty}\left(\frac{1}{2}b - \frac{1}{4}\sin(2b)\right) = \infty$. Thus, the integral diverges.

(g) $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Integrand has infinite discontinuity (vertical asymptote) at $x = 0$.

$$\lim_{b\to 0^+}\int_b^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{b\to 0^+} 2e^{\sqrt{x}}\Big|_b^1 = \lim_{b\to 0^+}(2e - 2e^{\sqrt{b}}) = 2e - 2.$$

The integral converges to $2e - 2$.

(e) $\displaystyle\int_{1/2}^1 \frac{2}{\sqrt{1-x^2}}\,dx$

Integrand has infinite discontinuity (vertical asymptote) at $x = 1$.

$$\lim_{b\to 1^-}\int_{1/2}^{b}\frac{2}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-} 2\arcsin b\Big|_{1/2}^{b} = \lim_{b\to 1^-}(2\arcsin b - 2\arcsin 1/2) = 2\frac{\pi}{2} - 2\frac{\pi}{6} = \frac{2\pi}{3}.$$

The integral converges to $\frac{2\pi}{3}$.

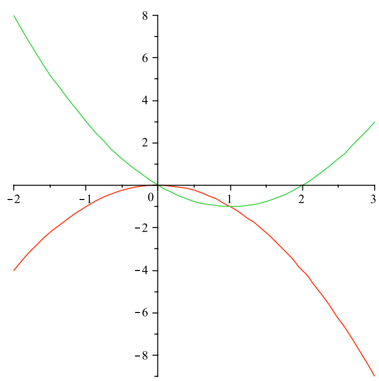
2. (i) Sketch the given curves and determine where they intersect.

   (ii) Find the area bounded by the given curves.

   Solution:

   (a) $y = -x^2$ and $y = x^2 - 2x$

   (i)

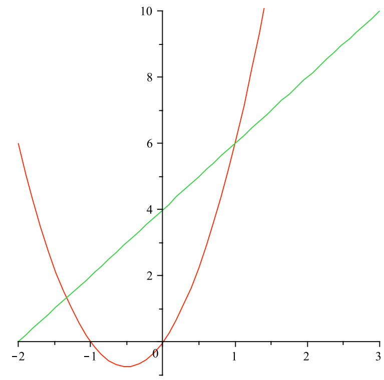


To find where the curves intersect, let $-x^2 = x^2 - 2x$. Then $2x^2 - 2x = 0$. Thus $x(x-1) = 0$. The curves intersect at $x = 0$ and $x = 1$.

(ii) Area $= \displaystyle\int_0^1 (-x^2 - (x^2 - 2x))\,dx = \int_0^1 (-2x^2 + 2x)\,dx = \left(-2\frac{x^3}{3} + x^2\right) = -\frac{2}{3} + 1 = \frac{1}{3}$.

(b) $y = 3x^2 + 3x$, $y = 2x + 4$, $x = -1$, and $x = 3$

(i)



To find where the curves intersect, let $3x^2 + 3 = 2x + 4$. The curves intersect at $x = -\dfrac{4}{3}$ and $x = 1$.

(ii) Area $= \displaystyle\int_{-1}^{1} (2x + 4 - (3x^2 + 3x))\, dx + \int_{1}^{3} (3x^2 + 3x - (2x + 4))\, dx =$

$$\int_{-1}^{1} (-3x^2 - x + 4)\, dx + \int_{1}^{3} (3x^2 + x - 4)\, dx = \left(-x^3 - \frac{x^2}{2} + 4x\right)\Bigg|_{-1}^{1} + \left(x^3 + \frac{x^2}{2} - 4x\right)\Bigg|_{1}^{3} =$$

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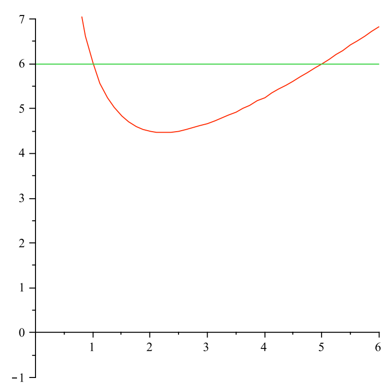
(c) $y = \dfrac{5}{x} + x$ and $y = 6$ (in the region $x > 0$)

(i)

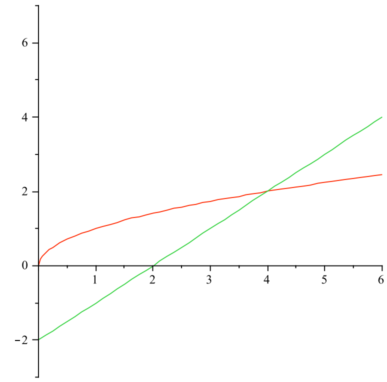


To find where the curves intersect, let $\dfrac{5}{x} + x = 6$. Thus $5 + x^2 - 6x = 0$. The curves intersect at $x = 1$ and $x = 5$.

(ii) Area $= \displaystyle\int_{1}^{5} \left(6 - (\frac{5}{x} + x)\right) dx = \left(6x - 5\ln|x| - \frac{x^2}{2}\right)\Bigg|_{1}^{5} = 12 - 5\ln 5$

(d) $y = \sqrt{x}$ and $y = x - 2$, $(x > 0)$

(i)

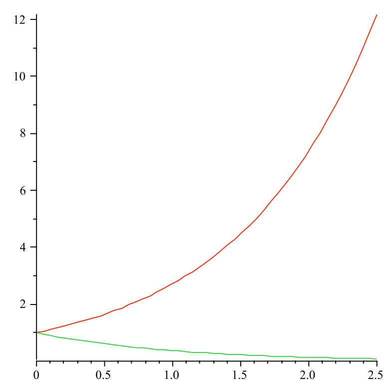


To find where the curves meet, let $\sqrt{x} = x - 2$. Square both sides (but keep in mind that $x > 0$). Then $x = x^2 - 4x + 4$. Thus $x = 1$ and $x = 4$. However, when $x = 1$, we have $y = x - 2 = -1$ and this point is not on the curve $y = \sqrt{x}$. Thus, the only point of intersection is when $x = 4$.

(ii) Area $= \displaystyle\int_0^4 (\sqrt{x} - (x-2))\, dx = \left(\frac{x^{3/2}}{3/2} - \frac{x^2}{2} + 2x\right)\Big|_0^4 = \frac{16}{3}.$

(e) $y = e^x$, $y = e^{-x}$, $x = 0$, and $x = 1$

(i)

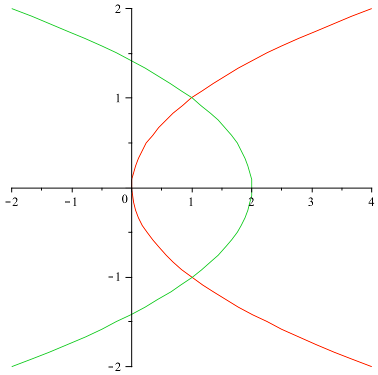


To find where the curves meet, let $e^x = \dfrac{1}{e^x}$. Thus $e^{2x} = 1$. Therefore $2x = 0$, i.e, $x = 0$.

(ii) Area $= \displaystyle\int_0^1 (e^x - e^{-x})\, dx = (e^x + e^{-x})\big|_0^1 = e + \frac{1}{e} - 2$

(f) $x = 2 - y^2$ and $x = y^2$

(i)



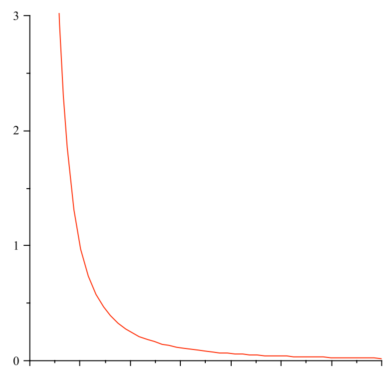
To find where the curves meet, let $2 - y^2 = y^2$. Thus $2y^2 = 2$, i.e., $y = \pm 1$.

(ii) Area $= \displaystyle\int_{-1}^{1} (2 - y^2 - y^2)\, dy = (2y - 2\frac{y^3}{3})\Big|_{-1}^{1} = \frac{8}{3}$.

3. Consider the curve $y = \dfrac{1}{x^2}$ for $1 \le x \le 6$.

Solution:

(a) Calculate the area under this curve on the given interval.



Area $= \displaystyle\int_1^6 \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_1^6 = \frac{5}{6}$.

(b) Determine c so that the line $x = c$ bisects the area of part (a). (*I.e.*, it divides the region into two regions of equal area.)

We need to find c between 1 and 6 such that $\displaystyle\int_1^c \frac{1}{x^2}\, dx = \int_c^6 \frac{1}{x^2}\, dx$. This means we need to find c such that $\displaystyle\int_1^c \frac{1}{x^2}\, dx = \frac{5}{12}$. We have

$\displaystyle\int_1^c \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_1^c = -\frac{1}{c} + 1$. We solve for c in $-\dfrac{1}{c} + 1 = \dfrac{5}{12}$. Thus $c = \dfrac{12}{7}$.

(c) Determine d so that the line $y = d$ bisects the area of part (a).

Since the line $y = d$ is a horizontal line, we will view the curve above as given by the equation $x = \dfrac{1}{\sqrt{y}}$. The area in part (a) is made up of the area of a rectangle of base 5 and height $\dfrac{1}{36}$ and the area between the horizontal line $y = 1/36$ and the graph of the function $y = \dfrac{1}{x^2}$. Since the area of the rectangle is $5/36$, which is smaller that $5/12$, we know that $d > 1/36$. Therefore, we need to find d such that $\displaystyle\int_d^1 \left(\frac{1}{\sqrt{y}} - 1\right) dy = \frac{5}{12}$. Since $\displaystyle\int_d^1 \left(\frac{1}{\sqrt{y}} - 1\right) dy = (2\sqrt{y} - y)\Big|_d^1 = 1 - 2\sqrt{d} + d$, we have $1 - 2\sqrt{d} + d = 5/12$ and thus $2\sqrt{d} = d + 7/12$. By squaring both sides we get $4d = d^2 + \dfrac{7}{6}d + \dfrac{7^2}{12^2}$. Solving $d^2 - \dfrac{17}{6}d + \dfrac{7^2}{12^2} = 0$, we obtain $d = \dfrac{17}{12} \pm \dfrac{1}{3}\sqrt{15}$. Sinde $d < 1$, we have $d = \dfrac{17}{12} - \dfrac{1}{3}\sqrt{15}$.