Math 132: Calculus for the Physical & Life Sciences 2
Problem Set 4
Due Friday, February 15, 2008, at the beginning of class.

General Directions: You must show all work for credit on these problems.

1. Use trigonometric substitutions (possibly after completing the square) to evaluate each of the following integrals.

a. $\int \frac{1}{\sqrt{9+4x^2}}\,dx$

Solution: The integral is $\frac{1}{3}\int \frac{dx}{\sqrt{1+(2x/3)^2}}$. Let $2x/3 = \tan\theta$, so $\sqrt{1+(2x/3)^2} = \sec\theta$ and $dx = (3/2)\sec^2(\theta)\,d\theta$. The integrand becomes

$$\begin{aligned} \frac{1}{2}\int \frac{\sec^2\theta\ d\theta}{\sec\theta} &= \frac{1}{2}\int \sec\theta\ d\theta \\ &= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C \\ &= \frac{1}{2}\ln|\sqrt{9+4x^2}/3 + 2x/3| + C, \end{aligned}$$

which can also be written as

$$\frac{1}{2}\ln|\sqrt{9+4x^2} + 2x| + C$$

(by incorporating $-\ln(3)/2$ into the constant of integration).

b. $\int \frac{\sqrt{x^2-1}}{x^2}\,dx$

Solution: Let $x = \sec\theta$, so $dx = \sec\theta\tan\theta\,d\theta$. The integrand becomes

$$\begin{aligned} \int \frac{\tan\theta\sec\theta\tan\theta\,d\theta}{\sec^2\theta} &= \int \frac{\tan^2\theta\,d\theta}{\sec\theta} \\ &= \int \frac{\sec^2\theta - 1\,d\theta}{\sec\theta} \\ &= \int \sec\theta - \cos\theta\,d\theta \\ &= \ln|\sec\theta + \tan\theta| - \sin\theta + C \\ &= \ln|x + \sqrt{x^2-1}| - \frac{\sqrt{x^2-1}}{x} + C \end{aligned}$$

c. $\displaystyle\int \frac{1}{\sqrt{3-2x-x^2}}\,dx$

Solution: Completing the square, $3-2x-x^2 = 4-(x+1)^2$. So we let $x+1 = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$. The integrand becomes

$$\begin{aligned}\int \frac{2\cos\theta\,d\theta}{2\cos\theta} &= \int d\theta \\ &= \theta + C \\ &= \sin^{-1}\left(\frac{x+1}{2}\right) + C.\end{aligned}$$

2. Integrate using the partial fraction technique (divide the denominator polynomial into the top first when necessary).

a. $\displaystyle\int \frac{1}{x^2+x-6}\,dx$

Solution: The denominator factors as $(x+3)(x-2)$, so the partial fractions are

$$\frac{1}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}.$$

This gives $1 = A(x-2) + B(x+3)$, so $B = 1/5$ and $A = -1/5$. Then

$$\int \frac{1}{x^2+x-6}\,dx = \int \frac{-1/5}{x+3} + \frac{1/5}{x-2}\,dx = \frac{-1}{5}\ln|x+3| + \frac{1}{5}\ln|x-2| + C$$

This could also be written as $\frac{1}{5}\ln\left|\frac{x-2}{x+3}\right| + C$ using properties of logarithms.

b. $\displaystyle\int \frac{x^4}{x^2+4}\,dx$

Since the degree of the numerator is larger than the degree of the denominator, we divide first, yielding $x^4 = (x^2-4)(x^2+4) + 16$. Therefore

$$\begin{aligned}\int \frac{x^4}{x^2+4}\,dx &= \int x^2 - 4\,dx + \int \frac{16\,dx}{x^2+4} \\ &= \frac{x^3}{3} - 4x + 8\tan^{-1}(x/2) + C\end{aligned}$$

c. $\displaystyle\int \frac{x+10}{2x^2+5x-3}\,dx$

Solution: The denominator factors as $(2x-1)(x+3)$, so the partial fractions are

$$\frac{x+10}{(2x-1)(x+3)} = \frac{A}{2x-1} + \frac{B}{x+3}.$$

This gives $x+10 = A(x+3) + B(2x-1)$, so $A = 3$ and $B = -1$. Then

$$\int \frac{x+10}{(2x-1)(x+3)}\,dx = \int \frac{3}{2x-1} + \frac{-1}{x+3}\,dx = \frac{3}{2}\ln|2x-1| - \ln|x+3| + C.$$

d. $\displaystyle\int \frac{x^2+1}{x^3+2x^2+x}\,dx$

Solution: The denominator factors as $x(x^2+2x+1) = x(x+1)^2$. Because of the repeated linear factor, the partial fractions here will be

$$\frac{x^2+1}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

After clearing denominators,

$$x^2+1 = A(x+1)^2 + Bx(x+1) + Cx.$$

Hence $A = 1$, $B = 0$ and $C = -2$. So the integral is

$$\int \frac{1}{x} - \frac{2}{(x+1)^2}\,dx = \ln|x| + \frac{2}{x+1} + C.$$

e. $\displaystyle\int \frac{x+4}{x^3+4x}\,dx$

Solution: The denominator factors as $x(x^2+4)$. Because of the quadratic factor, the partial fractions look like

$$\frac{x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4},$$

so after clearing denominators,

$$x+4 = A(x^2+4) + (Bx+C)x.$$

This gives $C = 1$, $A = 1$ and $B = -1$. So

$$\int \frac{x+4}{x^3+4x} = \int \frac{1}{x} + \frac{-x+1}{x^2+4}\,dx = \ln|x| - \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}(x/2) + C.$$

3. Integrate using the table of integrals in Stewart, and preliminary substitutions or algebraic manipulation as necessary. In each case, state clearly the number of the table entry you are using.

a. $\displaystyle\int \frac{x^2}{\sqrt{x^2+25}}\,dx$

Solution: Use number 26 in the table with $a = 5$. The answer is

$$\frac{x}{2}\sqrt{x^2+25} - \frac{25}{2}\ln|x + \sqrt{25+x^2}| + C.$$

b. $\displaystyle\int \sqrt{2x - x^2}\, dx$

Solution: Complete the square inside the radical: $\sqrt{2x - x^2} = \sqrt{1 - (x-1)^2}$. So after the substitution $u = x - 1$, this is $\int \sqrt{1 - u^2}\, du$, which is number 30 in the table with $a = 1$. The integral is

$$\frac{x-1}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1) + C.$$

c. $\displaystyle\int \frac{1}{\sqrt{e^{2x} - 1}}\, dx$

Solution: First substitute $u = e^x$, so $du = e^x dx$ and $dx = du/e^x = du/u$. The form is $\int \frac{1}{u\sqrt{u^2-1}}\, du$, which is number 18 in the table with $a = 1$. The answer is

$$\sec^{-1}(e^x) + C.$$

d. $\displaystyle\int \frac{\cos(x)}{\sqrt{4 - \sin^2(x)}}\, dx$

Solution: First substitute $u = \sin x$, so $du = \cos x\, dx$. The form is $\int \frac{1}{\sqrt{4-u^2}}\, du$, which is number 16 in the table with $a = 2$. The integral is

$$\sin^{-1}\left(\frac{\sin(x)}{2}\right) + C.$$

e. $\displaystyle\int \frac{\sec^2(x)}{\tan^2(x) + 2\tan(x) + 2}\, dx$

Solution: The substitution $u = \tan(x)$, $du = \sec^2(x)\, dx$ converts this to $\int \frac{du}{u^2+2u+2} = \int \frac{du}{(u+1)^2+1}$, which is in the basic arctangent integral form for $v = u + 1$ (number 17 in the table). Answer is

$$\tan^{-1}(\tan(x) + 1) + C.$$

f. $\displaystyle\int e^x \sqrt{1 + e^{2x}}\, dx$

Solution: Substitute $u = e^x$, then $du = e^x dx$, so the form is $\int \sqrt{1 + u^2}\, du$, which is number 21 in the table with $a = 1$. Answer is

$$\frac{e^x}{2}\sqrt{1+e^{2x}} + \frac{1}{2}\ln\left|e^x + \sqrt{1+e^{2x}}\right| + C.$$