Math 132: Calculus for Physical and Life Sciences 2
Extra Credit Problem Set 10
Due Wednesday, April 30, no later than 5:00pm

General Directions: This is an optional problem set. You may submit solutions for some or all of the following problems. Any points you earn will count as extra credit on the problem set component of your semester average. You must show all work for credit on these problems.

1. The metal making up a rod has density $\rho(x) = 4 + \sin(x)$ at location x. The rod extends from $x = 0$ to $x = \frac{3\pi}{2}$ along the x axis. Determine the total mass and the location of the center of mass of the rod.

   Solution: The total mass is

$$M = \int_0^{3\pi/2} 4 + \sin(x)\, dx = 4x - \cos(x)\Big|_0^{3\pi/2} = 6\pi + 1.$$

   The moment with respect to the y-axis is

$$M_y = \int_0^{3\pi/2} x(4 + \sin(x))\, dx.$$

   For this, we expand out and integrate $\int x \sin(x)\, dx$ by parts with $u = x$, $dv = \sin(x)\, dx$. This gives

$$\begin{aligned} M_y &= \int_0^{3\pi/2} x(4 + \sin(x))\, dx \\ &= \int_0^{3\pi/2} 4x + x\sin(x)\, dx \\ &= 2x^2 - x\cos(x) + \sin(x)\Big|_0^{3\pi/2} \\ &= \frac{9\pi^2}{2} - 1. \end{aligned}$$

   The center of mass is located at

$$\overline{x} = \frac{M_y}{M} = \frac{\frac{9\pi^2}{2} - 1}{6\pi + 1} \doteq 2.187.$$

   (As might be expected from the form of the density function, this is located to the left of the midpoint of the wire.)

2. A metal plate with constant density 5 gm/cm^2 has the shape of the region bounded by $y = \sqrt{x^2 + 1}$ and the x-axis, for $0 \le x \le 4$. Find the location of the center of mass of the plate.

Solution: The total mass is

$$M = \int_0^4 5\sqrt{x^2+1}\,dx.$$

For this integral, we can either use the tangent substitution $x = \tan\theta$ or consult the table of integrals. This is the form of #21, with $u = x$ and $a = 1$. The value is

$$M = \frac{5x}{2}\sqrt{1+x^2} + \frac{5}{2}\ln(x+\sqrt{1+x^2})\Big|_0^4 = 10\sqrt{17} + \frac{5}{2}\ln(4+\sqrt{17}).$$

The moment with respect to the y-axis then given by an integral we can evaluate with the substitution $u = 1 + x^2$, $du = 2x\,dx$:

$$M_y = \int_0^4 5x\sqrt{1+x^2}\,dx = \frac{5}{2}\int_{u=1}^{u=17} u^{1/2}\,du = \frac{5}{3}(17\sqrt{17}-1).$$

Finally, the moment with respect to the x-axis is

$$M_x = \int_0^4 5\cdot\frac{1}{2}(\sqrt{1+x^2})^2\,dx = \frac{5}{2}\int_0^4 1 + x^2\,dx = \frac{190}{3}.$$

Hence the center of mass is the point with coordinates

$$\begin{aligned}\overline{x} &= \frac{M_y}{M} \doteq 2.478,\\ \overline{y} &= \frac{M_x}{M} \doteq 1.363.\end{aligned}$$

3. The probability of a certain type of transistor failing between $t = a$ and $t = b$ (months), for $0 < a < b$, is given by

$$P(a \le t \le b) = c\int_a^b e^{-ct}\,dt.$$

(a) If the probability of failure within the first six months is .10, what is the value of c?

Solution: (Note: The function

$$f(t) = ce^{-ct}, \quad \text{for} \quad t > 0$$

and $f(t) = t$ for $t \le 0$ is the pdf.) The given information says

$$.1 = \int_0^6 ce^{-ct}\,dt = -e^{-ct}\big|_0^6 = 1 - e^{-6c}.$$

So $c = \frac{ln(.9)}{-6} \doteq .01756$.

(b) Given the value of c in part (a), what is the probability that the transistor lasts at least 6 months?

Solution: This is the same as the probability that the time to failure is a number ≥ 6 months. That probability is computed by

$$\int_6^\infty ce^{-ct}\,dt = 1 - \int_0^6 ce^{-ct}\,dt = 1 - .1 = .9.$$

(c) Given the value of c in part (a), what is the *mean* life of this type of transistor?

Solution: The mean life is

$$\begin{aligned}
\bar{t} &= \int_0^\infty tce^{-ct}\,dt \\
&= \lim_{b\to\infty} \int_0^b tce^{-ct}\,dt \\
&= \lim_{b\to\infty} -te^{-ct} - \frac{1}{c}e^{-ct}\Big|_0^b \qquad \text{(using parts: } u = t,\ dv = ce^{-ct}) \\
&= \lim_{b\to\infty} \frac{-b}{e^{bc}} - \frac{1}{ce^{bc}} + \frac{1}{c} \\
&= \frac{1}{c} \\
&\doteq 56.95\,\text{months}.
\end{aligned}$$

(d) Given the value of c in part (a), what is the *median* life of this type of transistor?

Solution: The cumulative distribution is given by $F(t) = 1 - e^{-ct}$ if $t > 0$ and $F(t) = 0$ if $t \leq 0$. Hence

$$\frac{1}{2} = F(t) = 1 - e^{-ct} \Rightarrow e^{-ct} = \frac{1}{2} \Rightarrow t = \frac{\ln(1/2)}{-c} = 39.37\,\text{months}.$$

4. Suppose that t is the time (in hours) it takes for calculus students to complete their final exam. Assume that all students finish within 3 hours and that the probability density function for the time t is

$$f(t) = \begin{cases} \dfrac{4x^3}{81} & \text{if } 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) What proportion of the students take between 1.5 and 2.5 hours to complete the exam?

Solution: This is the same as the probability that the time is between 1.5 and 2.5:

$$P(1.5 \leq x \leq 2.5) = \int_{1.5}^{2.5} \frac{4x^3}{81}\,dx = \frac{x^4}{81}\Big|_{1.5}^{2.5} \doteq .4198.$$

About 42% of the students will take between 1.5 and 2.5 hours.

(b) What is the *mean time* for students to complete the exam?

Solution: The mean time is

$$\begin{aligned}
\overline{x} &= \int_0^3 x \cdot \frac{4x^3}{81}\,dx && (1)\\
&= \int_0^3 \frac{4x^4}{81}\,dx && (2)\\
&= \left.\frac{4x^5}{405}\right|_0^3 && (3)\\
&= \frac{12}{5} = 2.4\,\text{hours.} && (4)
\end{aligned}$$

(c) What is the *median time* for students to complete the exam?

Solution: The cdf is the antiderivative of $\frac{4x^3}{81}$ which takes the value 0 at 0, so $F(x) = \frac{x^4}{81}$. We have $\frac{1}{2} = \frac{x^4}{81}$ when $x = \left(\frac{81}{2}\right)^{1/4} \doteq 2.523$ hours.

(d) At what time should the (evil) professor set the end of the exam period if he wants to make sure that only 2/3 of the students have completed the exam when the papers are collected?

Solution: We need to solve $F(x) = \frac{2}{3}$. This gives $x = (54)^{1/4} \doteq 2.71$ hours (about 2 hours and 43 minutes).

5. The distribution of scores on IQ exams is often modeled by a *normal* distribution with mean $\mu = 100$ and standard deviation $\sigma = 15$.

(a) Give the formula for the normal pdf that fits this description.

Solution: The appropriate normal pdf is

$$f(x) = \frac{1}{\sqrt{450\pi}} e^{-(x-100)^2/450}.$$

(a) Estimate the fraction of the population with IQ scores between 115 and 120 by applying a Midpoint Riemann sum approximation for the appropriate integral. Use $n = 5$ subintervals in the Riemann sum.

Solution: This is the same as the probability that a randomly chosen individual has IQ in this interval:

$$P(115 \le x \le 120) = \int_{115}^{120} f(x)\,dx \doteq 0.06743971593$$

(about 6.7% of the population have IQ scores in this range).

(b) Estimate the fraction of the population with IQ scores between 140 and 150 by the same method as in part (b).

Solution:

$$P(140 \le x \le 150) = \int_{140}^{150} f(x)\,dx \doteq 0.00338$$

(about 0.3% of the population).

6. Let μ and $\sigma^2 > 0$ be any two real constants.

   (a) Show that the normal pdf

   $$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

   has exactly one critical number at $x = \mu$, and that $f(x)$ has a local maximum at $x = \mu$.

   Solution: See below.

   (b) Show that $f(x)$ has inflection points at $x = \mu \pm \sigma$.

   Solution: The constant $\frac{1}{\sqrt{2\pi\sigma^2}}$ in the formula for the normal density is irrelevant for the location of the critical number(s) and the inflection points, so we will ignore it and consider the following function $g(x)$ and its derivatives:

   $$\begin{aligned} g(x) &= e^{-(x-\mu)^2/(2\sigma^2)} \\ \Rightarrow g'(x) &= -e^{-(x-\mu)^2/(2\sigma^2)} \cdot (x-\mu)/\sigma^2 \quad \text{(chain rule)} \\ \Rightarrow g''(x) &= -e^{-(x-\mu)^2/(2\sigma^2)} \left(\frac{1}{\sigma^2} - \frac{(x-\mu)^2}{\sigma^4} \right) \quad \text{(product rule).} \end{aligned}$$

   We have $g'(x) = 0$ if and only if $x = \mu$ which shows the first part of (a). By the second derivative test,

   $$g''(\mu) = -\frac{1}{\sigma^2} < 0,$$

   which implies that $g(x)$ and $f(x)$ have maxima at $x = \mu$. We have $g''(x) = 0$ if and only if

   $$\begin{aligned} 0 &= \frac{1}{\sigma^2} - \frac{(x-\mu)^2}{\sigma^4} \\ \Rightarrow (x-\mu)^2 &= \sigma^2 \\ \Rightarrow x - \mu &= \pm\sigma \\ \Rightarrow x &= \mu \pm \sigma, \end{aligned}$$

   which is what we wanted to show.

   (c) Give qualitative sketches of $y = f(x)$ with $\mu = 4$ and $\sigma^2 = 4$, and the cumulative distribution function for this normal distribution.

Solution: The graph $y = f(x)$ has a maximum at $x = 4$, inflections at $x = 4 \pm 2 = 2, 6$. The cumulative distribution $F(x)$ is increasing for all x and has a graph that is an "S"-shaped curve with $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$. It has maximum positive slope at x 4.