College of the Holy Cross, Spring 2008
Solutions for Math 132, Midterm Exam 3 (All Sections)
Wednesday, April 23, 7 PM

I.

A. [10 points] Circle the number of the plot showing the direction field for each of the following differential equations. (Note that there are only 3 plots, so the correct answer for one is "None.")

Solution:

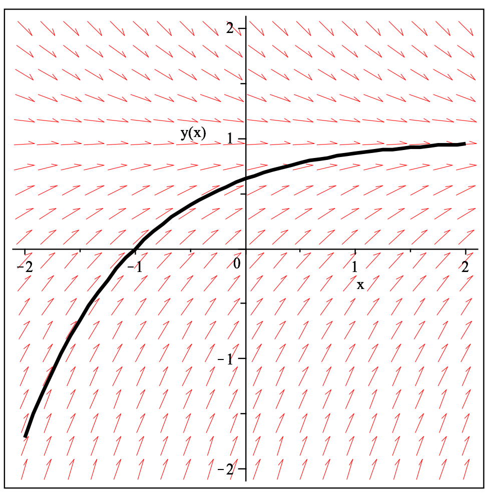
(1) $y' = x$ is plot III (note negative slopes for all $x < 0$ and positive slopes for all $x > 0$)

(2) $y' = 1 + y^2$ matches None of the plots (by process of elimination, or by noting that none of the direction fields has positive slope everywhere)

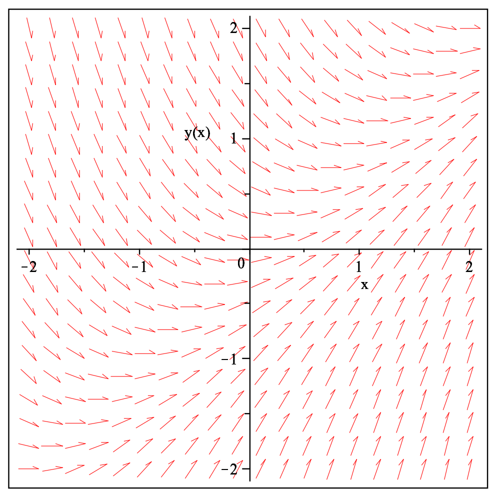
(3) $y' = 1 - y$ is plot I (note zero slope along the line $y = 1$)

(4) $y' = x - y$ is plot II (note zero slope along the line $y = x$)

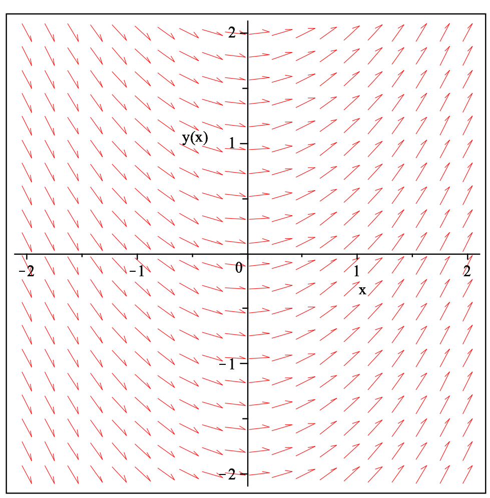
I:



II:



III:



B. [5 points] On the plot for the equation $y' = 1 - y$ from (3) of part A, give a qualitative sketch of the graph of the solution satisfying the initial condition $y(-1) = 0$. Show as much of the graph as you can for both positive and negative x.

Solution: See plot I above.

II. All parts of this problem deal with the differential equation $y' = 7 - y$.

A. [4 points] Use 4 steps of Euler's method to approximate the solution of this equation with the initial condition $y(0) = 4$ at $x = 2$.

Solution: With $x_0 = 0$ and $x_4 = 2$, we will use $\Delta x = \frac{2-0}{4} = \frac{1}{2}$. The steps of Euler's Method are

$$\begin{aligned}
y_1 = y_0 + (7 - y_0)\Delta x &= 4 + (3)(.5) = 5.5 \\
y_2 = y_1 + (7 - y_1)\Delta x &= 5.5 + (1.5)(.5) = 6.25 \\
y_3 = y_2 + (7 - y_2)\Delta x &= 6.25 + (.75)(.5) = 6.625 \\
y_4 = y_3 + (7 - y_3)\Delta x &= 6.625 + (.375)(.5) = 6.8125
\end{aligned}$$

The approximate value of $y(2)$ is $y(2) \doteq y_4 = 6.8125$.

B. [6 points] Find the general solution $y(x)$ of the equation by separating variables and integrating.

Solution: We separate variables, integrate, then exponentiate to solve for y:

$$\begin{aligned}
\frac{dy}{7 - y} &= dx \\
\int \frac{dy}{7 - y} &= \int dx \\
-\ln|7 - y| &= x + c \\
7 - y &= be^{-x} \quad \text{where } b = \pm e^{-c} \text{ is another arbitrary constant} \\
y &= 7 - be^{-x}.
\end{aligned}$$

C. [5 points] Find the particular solution $y(x)$ satisfying the initial condition $y(0) = 4$ and compute the exact value of $y(2)$.

Solution: Substituting $x = 0$ and $y = 4$ gives $4 = 7 - b$, so $b = 3$. The particular solution is $y = 7 - 3e^{-x}$, and $y(2) = 7 - 3e^{-2} \doteq 6.59399$. The approximation given by Euler's Method in part A. is an overestimate.

III. All parts of this question deal with the infinite series

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots \ .$$

A. [4 points] Find a general formula for the nth term of the series as a function of n and write the series in summation ("sigma") notation.

Solution: If we index the first term above with $n = 1$, the second with $n = 2$, etc., then the general term is $a_n = \frac{n}{n+1}$, and the series is $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$.

B. [4 points] Call the nth term in the series a_n. What is $\lim_{n\to\infty} a_n$?

Solution: We have

$$\begin{aligned} \lim_{n\to\infty} \frac{n}{n+1} &= \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \\ &= 1. \end{aligned}$$

(This limit could also be computed using L'Hopital's Rule.)

C. [4 points] Write out the first 3 *partial sums* of the series.

Solution: The first three partial sums are

$$\begin{aligned} s_1 &= \frac{1}{2} \\ s_2 &= \frac{1}{2} + \frac{2}{3} = \frac{7}{6} \\ s_3 &= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{23}{12}. \end{aligned}$$

D. [3 points] Does this series converge or diverge? Explain your answer.

Solution: Since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, this series *diverges* by the Test for Divergence.

IV.

A. [5 points] Does the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n e^n}{\pi^n}$ converge or diverge?

Solution: This series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n e^n}{\pi^n} = \sum_{n=0}^{\infty} \left(\frac{-e}{\pi}\right)^n$$

is a *geometric series* with first term $a = 1$ and ratio $r = -e/\pi$. Since $|r| < 1$, the series is convergent and

$$\sum_{n=0}^{\infty} \left(\frac{-e}{\pi}\right)^n = \frac{1}{1 - \frac{-e}{\pi}} = \frac{\pi}{\pi + e}.$$

B. [10 points] Explain why the Integral Test can be applied to the series $\sum_{n=1}^{\infty} \frac{n}{e^{3n}}$ and use it to determine if the series converges or diverges.

Solution: The function $f(x) = xe^{-3x}$ gives the terms in this series when we substitute $x = n$ for $n = 1, 2, 3, \ldots$. The function $f(x)$ is continuous for all x, and $f(x) > 0$ for all $x > 0$. Moreover,

$$f'(x) = -3xe^{-3x} + e^{-3x} = e^{-3x}(1 - 3x) < 0$$

for all $x \geq \frac{1}{3}$. Therefore $f(x)$ is positive and decreasing for $x \geq 1$ and the Integral Test applies. We integrate by parts with $u = x$, $dv = e^{-3x}$ to compute

$$\begin{aligned}
\int_1^{\infty} xe^{-3x}\,dx &= \lim_{b\to\infty} \left.\frac{-1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right|_1^b \\
&= \lim_{b\to\infty} \frac{-b}{3e^{3b}} - \frac{1}{9e^{3b}} + \frac{1}{3e^3} + \frac{1}{9e^3} \\
&= \frac{4}{9e^3}.
\end{aligned}$$

Since the improper integral converges, the series also converges.

V. All parts of this question refer to the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{\sqrt{n}}.$$

A. [9 points] Use the Ratio Test to determine the radius of convergence.

Solution: Applying the Ratio Test,

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x-3)^n} \right| &= \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}\,|x-3| \\
&= |x-3|.
\end{aligned}$$

For absolute convergence, we need $|x - 3| < 1$, so $2 < x < 4$. The series is centered at $a = 3$, so the radius of convergence is 1.

B. [6 points] Test convergence at the endpoints of the interval from part A to determine the interval of convergence. Explain your conclusions.

Solution: At $x = 2$, we substitute and obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

This is an alternating series and $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ with $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. The Alternating Series Test implies that this is a convergent series.

At $x = 4$, we have

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}.$$

This is a p-series with $p = 1/2$. Since $p < 1$, the series diverges there. The interval of convergence is $[2, 4)$.

VI. [10 points] Find the Taylor polynomial of degree $n = 3$ for $f(x) = 2 + 3x + x^3$ at $a = 1$.

Solution: We have

$$\begin{aligned} f(1) &= 6 \\ f'(x) = 3 + 3x^2 \Rightarrow f'(1) &= 6 \\ f''(x) = 6x \Rightarrow f''(1) &= 6 \\ f'''(x) = 6 \Rightarrow f'''(1) &= 6. \end{aligned}$$

Therefore the Taylor polynomial is

$$p_3(x) = 6 + 6(x-1) + \frac{6}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3 = 6 + 6(x-1) + 3(x-1)^2 + (x-1)^3.$$

(If you expand this out and collect powers of x, you will see that $p_3(x) = f(x)$.)

VII.

A. [5 points] Starting from the Taylor series for $\sin(x)$ at $a = 0$, find a series representation for $f(x) = \dfrac{\sin(x) - x}{x^3}$. Give the first three nonzero terms in your series.

Solution: Since

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

we have

$$\frac{\sin(x) - x}{x^3} = \frac{1}{x^3}\left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = \frac{-1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \cdots.$$

B. [5 points] Use your answer in part A to determine $\lim_{x\to 0} \dfrac{\sin(x) - x}{x^3}$.

Solution:

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} = \lim_{x\to 0} \frac{-1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} = \frac{-1}{6}.$$

C. [5 points] Use your answer in part A to determine the first three nonzero terms in a series representing the function

$$F(x) = \int_0^x \frac{\sin(t) - t}{t^3}\, dt.$$

Solution: We integrate the series from part A term by term to obtain

$$\int_0^x \frac{-1}{3!} + \frac{t^2}{5!} - \frac{t^4}{7!} + \cdots\, dt = \frac{-x}{3!} + \frac{x^3}{3 \cdot 5!} - \frac{x^5}{5 \cdot 7!} + \cdots.$$

A summation notation form would be

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1) \cdot (2n+3)!}.$$