Mathematics 132 – Calculus for Physical and Life Sciences 2
Lab Day 4: Visualizing Convergence of Taylor Series, Approximations
April 11, 2008 – Lab Writeups due in class on Monday, April 21.

Goals

In today's lab will use Maple to compute Taylor polynomials, and use graphical and numerical methods to understand the *convergence* of Taylor series.

Background

Recall from class on Wednesday that if f is a function that has derivatives of all orders, then its *Taylor series* at $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and provided that this series converges on some interval, it defines a function whose nth derivative at $x = a$ is the same as the nth derivative of $f(x)$ for all n. If we take the partial sums of the Taylor series, we obtain *Taylor polynomials.* There is exactly one polynomial $p_k(x)$ of degree $\leq k$ whose first k derivatives at $x = a$ are the same as the corresponding derivatives of f at $x = a$. The *Taylor polynomials* of f at $x = a$ are computed by the formula:

$$p_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Taylor Polynomials in Maple

Maple has a "built-in" function called `taylor` that can be used to compute Taylor polynomials of functions. The general format is

```
taylor(f(x),x=a,d);
```

where $f(x)$ is the function to be approximated, a is the x-value where the Taylor polynomials will be expanded, and $d \geq 1$ is an integer.

For example, try entering the following command which computes the 5th degree Taylor polynomial for $f(x) = e^x$ at $a = 0$

```
taylor(exp(x),x=0,6);
```

Note two things:

(1) The output

$$1 + x + (1/2)x^2 + (1/6)x^3 + (1/24)x^4 + (1/120)x^5 + O(x^6)$$

is a polynomial, plus another term. The other term – $O(x^6)$ – is intended to describe the size of the error if we use this polynomial to approximate e^x. The way to interpret this is that the error will go to zero like (a constant times) x^6 (at least) in the limit as $x \to 0$. To get rid of the error term, you can "nest" the `taylor` command inside a `convert` command like this:

```
convert(taylor(exp(x),x=0,6),polynom);
```

Try this and note the output. *You should use this convert step every time you want just the Taylor polynomial, without the error term.*

(2) The 6 in the Taylor command is *one more than* the degree of the polynomial. To get the nth degree polynomial, you will *always want to take* $d = n + 1$ in the Maple command. *This is one case where you don't want to ask "why?" Just accept that this is the way Maple is set up!*

Lab Questions

(A) In this question, you will generate plots of $cos(x)$, together with its Taylor polynomials of degrees $n = 2, 4, 6, 8, 10$ at $a = 0$ and visualize what is happening as more and more terms are added into the Taylor polynomials.

(1) First plot $\cos(x)$ and it Taylor polynomial of degree 2 together on the same axes with $-\pi \leq x \leq \pi$. You can use these commands, for instance, first to compute the Taylor polynomial, assign it the name `p2`, then plot it with the cosine function:

```
p2 := convert(taylor(cos(x),x=0,3),polynom);
plot([cos(x), p2],x=-Pi..Pi,color=[blue,red]);
```

Notes:

a. The $\cos(x)$ will be the blue graph and the polynomial will be the red graph – the colors match the order in the list of functions.

b. There is no (x) after the `p2` in the plotting command; we computed the Taylor polynomial as an expression in the command before, assigned it to the symbolic variable $p2$ and then used that expression in the plot command.

c. *When you hand in your lab write-ups, since you will probably be printing out your work on a black and white printer, please put in labels by hand showing which graph is which.*

(2) Now plot the absolute error function: $|\cos(x) - p_2(x)|$. (In Maple, the absolute value function is called `abs`.) What is the largest difference between $\cos(x)$ and $p_2(x)$ on this interval?

(3) Repeat part (1) for the Taylor polynomial of degree 4 of $\cos(x)$ at $a = 0$.

(4) Does using the fourth degree polynomial to approximate $\cos(x)$ seem to yield better results than using the polynomial of degree 2? For instance, is the absolute error for this polynomial smaller than the absolute error for $p_2(x)$ on the whole interval?

(5) Repeat parts (1) and (2) for the Taylor polynomials of degree 6, 8, 10 of $\cos(x)$ at $a = 0$. When you graph the polynomials of degree 8 and 10 together with the cosine function, you may actually only see one graph on the interval $-\pi \leq x \leq \pi$. What does that mean? Try plotting a bigger range of x-values until you see the graphs start

to "split apart." When does that happen for the degree 8 polynomial? When for the degree 10 polynomial?

(B) Now, consider the function $f(x) = \ln(1+x)$, expanding around $a = 0$ again.

(1) Plot the Taylor polynomials of degrees 1, 2, 3, 4, 5 and the function on the interval $-1 < x < 1$.
(2) For each degree, find an interval where the Taylor polynomial of that degree approximates $f(x)$ to within 10^{-3} – that is: $|f(x) - p_n(x)| < 10^{-3}$ for all x in the interval. How do the intervals change as n increases? For instance, does it seem that the interval will grow to arbitrary length as n increases, or is there a "limit" to its size on one end or the other?

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