Math 132: Calculus for the Physical & Life Sciences 2
Lab Day 1: Understanding Errors for Approximate Integration
Due Friday, February 22, 2008, at the beginning of class.

Background and Goals

We have discussed various methods for approximating definite integrals: left- and right-hand Riemann sums (L_n, R_n), midpoint Riemann sums (M_n), the Trapezoidal Rule (T_n), and Simpson's Rule (S_n). The main goal of this lab is to study how the *error* for each of these approximation methods:

$$\text{Error} = \text{Exact Value} - \text{Approximate Value}.$$

depends on the number of subdivisions n used to construct the approximation. We also want to understand why the weighted average form of Simpson's Rule:

$$S_{2n} = \frac{2}{3}M_n + \frac{1}{3}T_n \tag{1}$$

gives an explanation for why Simpson's Rule appears to be so much more accurate than the other methods.

Maple Commands

The commands that you will need to use are `RiemannSum`, `ApproximateInt` from the Student[Calculus1] package, which you load by typing

```
with(Student[Calculus1]);
```

To enter a function, say $f(x) = x^3 - 2x^2 - 7x + 4$, type

```
f := x-> x^3-2*x^2-7*x+4;
```

To compute one of the Riemann sum approximations, say the midpoint sum over the interval $[-2, 4]$ with $n = 8$ subintervals use

```
evalf(RiemannSum(f(x),x=-2..4,method=midpoint,partition=8));
```

To compute one of the other approximations, say the Trapezoid Rule over the interval $[-2, 4]$ with $n = 8$ subintervals, type

```
evalf(ApproximateInt(f(x),x=-2..4,method=trapezoid,partition=8));
```

To plot the graph of a function over an interval, use the `plot` command.

```
plot(f(x),x=-2..4);
```

A Comment

As we discussed in class on Wednesday, in Maple, the command `ApproximateInt` with the `method=simpson` and `partition=n` options actually corresponds to the Simpson's Rule approximation S_{2n} the way we wrote down the formula to start in class. Since the weighted average form (1) of Simpson's Rule relates S_{2n} to M_n and T_n, though, *we will always take equal partition numbers (n values) for all three methods when making comparisons.*

Lab Questions

I. You will first work with the definite integral

$$\int_0^1 x^2\sqrt{4-x^2}\,dx = \frac{\pi}{3} - \frac{\sqrt{3}}{4} \approx 0.6141848494$$

In Maple, π is entered as `Pi` and the command `sqrt` is the square root function.

A) (Do before or after the lab.) Verify the exact value of the integral above, using a trig substitution.

B) Complete Table A on the accompanying sheet for the function $f(x) = x^2\sqrt{4-x^2}$ over $[0, 1]$. (Compute the values using Maple, and fill in the table by hand, recording *all* decimal places Maple displays.)

C) For each method, when you double n, the error should be (approximately) divided by 2^p for some power p. Find that power for each of the methods. Put your results in the Table B on the accompanying page.

D) For each method the error can be estimated as

$$\boxed{E(n) \approx \frac{k}{n^p}}$$

where p is the power in the table above and k is some constant. In each method, determine the effect on the error of multiplying n by 10 and put your results in Table C.

E) Take $n = 16$. How do the errors for the Midpoint Riemann sum M_n and the Trapezoidal rule T_n compare?

F) Based on your answer to part E, about how large would n have to be for M_n to be accurate to 10 decimal places (that is, the error for M_n to be less than $.5 \times 10^{-10}$). Compute M_n for your n and compare to the exact value.

II. Repeat question I for the integral

$$\int_0^1 e^{x^2}\,dx$$

(which cannot be evaluated by means of the Evaluation Theorem because $f(x) = e^{x^2}$ has no antiderivative in the class of elementary functions). How will you get an "exact" value to approximate the errors for the four methods for $n = 2, 4, 8, 16$?

III.

A) (Do before or after lab.) Explain why S_{2n} is always between M_n and T_n. *Hint:* Either $M_n \le T_n$ or $T_n \le M_n$. Consider each case separately.

B) Based on your answers to part A above and parts I E and II E, explain why it is reasonable that Simpson's Rule should produce much more accurate estimates for the same `partitions` number n than either the Midpoint Rule or the Trapezoidal rule for the same n. See the weighted average form (1) above.

Approximations for $\int_0^1 x^2\sqrt{4-x^2}\,dx$

Table A

n	2	4	8	16
L_n				
$E_{left}(n)$				
R_n				
$E_{right}(n)$				
M_n				
$E_{mid}(n)$				
T_n				
$E_{trap}(n)$				
S_{2n}				
$E_{simp}(n)$				

Table B

Method	LEFT	RIGHT	MID	TRAP	SIMP
p					

Table C

Method	LEFT	RIGHT	MID	TRAP	SIMP
Error is divided by:					

Approximations for $\int_0^1 e^{x^2}\,dx$

Table A

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Table B

Method	LEFT	RIGHT	MID	TRAP	SIMP
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Table C

Method	LEFT	RIGHT	MID	TRAP	SIMP
Error is divided by:					