**College of the Holy Cross, Spring Semester, 2008**
**MATH 132, Section 01, Solutions for Final Exam**
**Thursday, May 8, 2:30 PM**
Professor Little

I.

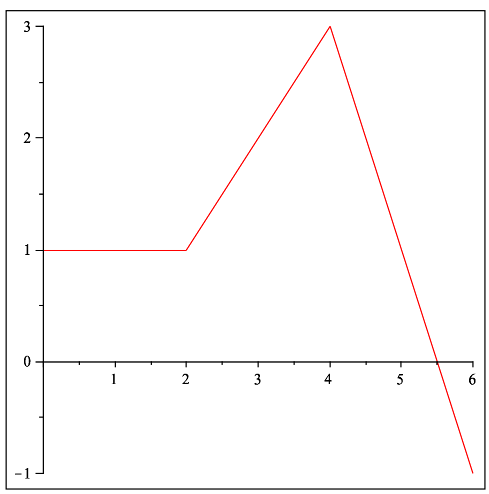
(A) (5) Compute the derivative of the function $g(x) = \displaystyle\int_0^{3x} \frac{\sin(t)}{t}\, dt$.

**Solution:** By the first part of the Fundamental Theorem of Calculus and the Chain Rule for derivatives:

$$g'(x) = \frac{\sin(3x)}{3x} \cdot 3 = \frac{\sin(3x)}{x}.$$

(B) (4) Let $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 2 \\ x - 1 & \text{if } 2 < x \le 4 \\ 11 - 2x & \text{if } 4 < x \le 6 \end{cases}$. Sketch the graph $y = f(x)$ on the axes provided below.

**Solution:**




(C) (6) Let $F(x) = \int_0^x f(t)\, dt$, where $f(t)$ is the function from part (B). Complete the following table of values for $F(x)$:

**Solution:** The value $F(x)$ represents the signed area between the graph $y = f(x)$ and the $x$-axis:

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $F(x)$ | 0 | 1 | 2 | 3.5 | 6 | 8 | 8 |

II. Compute the following integrals. Some of these may be forms covered by entries in the table of integrals. Half credit will be given for using a table entry; full credit only for showing all work leading to the final answer.

(A) (5) $\displaystyle\int \frac{x^4 - 3x^3 + \sqrt{x}}{x^{2/3}}\, dx$

**Solution:** Split into separate fractions, simplify and integrate:

$$\begin{aligned}\int \frac{x^4 - 3x^3 + \sqrt{x}}{x^{2/3}}\, dx &= \int x^{10/3} - 3x^{7/3} + x^{-1/6}\, dx \\ &= \frac{3}{13}x^{13/3} - \frac{9}{10}x^{10/3} + \frac{6}{5}x^{5/6} + C.\end{aligned}$$

(B) (5) $\displaystyle\int x\cos(x^2)\, dx$

**Solution:** Use a $u$-substitution with $u = x^2$, so $du = 2x\, dx$. This gives

$$\int x\cos(x^2)\, dx = \frac{1}{2}\int \cos(u)\, du = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin(x^2) + C.$$

(C) (10) $\displaystyle\int \frac{\sec^2(5x)\, dx}{(\tan(5x) + 7)^5}$

**Solution:** This one can also be handled by the $u$-substitution $u = \tan(5x) + 7$, for which $du = 5\sec^2(5x)\, dx$ by the Chain Rule. Then

$$\int \frac{\sec^2(5x)\, dx}{(\tan(5x) + 7)^5} = \frac{1}{5}\int u^{-5}\, du = \frac{-1}{20}u^{-4} + C = \frac{-1}{20}(\tan(5x) + 7)^{-4} + C.$$

(D) (10) $\displaystyle\int_1^e x^7 \ln(x)\, dx.$

**Solution:** Integrate by parts with $u = \ln(x)$ and $dv = x^7\, dx$. Then $du = \frac{1}{x}\, dx$ and $v = \frac{1}{8}x^8$ and by the integration by parts formula,

$$\int x^7 \ln(x)\, dx = \frac{x^8}{8}\ln(x) - \int \frac{1}{8}x^7\, dx = \frac{x^8}{8}\ln(x) - \frac{x^8}{64} + C.$$

For the definite integral, we apply the Evaluation Theorem to get

$$\int_1^e x^7 \ln(x)\, dx = \frac{x^8}{8}\ln(x) - \frac{x^8}{64}\bigg|_1^e = \frac{e^8}{8} - \frac{e^8}{64} + \frac{1}{64} = \frac{7e^8 + 1}{64}.$$

(E) (12.5) $\displaystyle\int \frac{x^2}{\sqrt{25 - x^2}}\, dx$

**Solution:** We use the trigonometric substitution $x = 5\sin\theta$, so $dx = 5\cos\theta\, d\theta$, and $\sqrt{25-x^2} = \sqrt{25(1-\sin^2\theta)} = 5\cos\theta$. This simplifies to

$$\int \frac{25\sin^2\theta}{5\cos\theta}\cdot 5\cos\theta\, d\theta = 25\int \sin^2\theta\, d\theta.$$

Now we use the half-angle identity to integrate the even power of $\sin\theta$, then convert back to $x$:

$$\begin{aligned} 25\int \sin^2\theta\, d\theta &= 25\int \frac{1}{2}(1-\cos(2\theta))\, d\theta \\ &= \frac{25}{2}\theta - \frac{25}{4}\sin(2\theta) + C \\ &= \frac{25}{2}\theta - \frac{25}{2}\sin\theta\cos\theta + C \\ &= \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right) - \frac{1}{2}x\sqrt{25-x^2} + C. \end{aligned}$$

(F) (12.5) $\displaystyle\int \frac{x}{(x+1)(x^2+1)}\, dx$

**Solution:** By partial fractions,

$$\frac{x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}.$$

Clearing denominators,

$$x = A(x^2+1) + (Bx+C)(x+1),$$

so equating coefficients, $A+B=0$, $B+C=1$, and $A+C=0$. Solving simultaneously,

$$A = \frac{-1}{2}, \qquad B = \frac{1}{2}, \qquad C = \frac{1}{2}.$$

Then

$$\begin{aligned} \int \frac{x}{(x+1)(x^2+1)}\, dx &= \int \frac{-1/2}{x+1} + \frac{x/2+1/2}{x^2+1}\, dx \\ &= \frac{-1}{2}\ln|x+1| + \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + C. \end{aligned}$$

III.

(A) (5) Use a midpoint Riemann sum with $n=4$ to approximate $\displaystyle\int_0^2 e^{x^2}\, dx$.

**Solution:** The midpoint Riemann sum approximation is

$$\int_0^2 e^{x^2}\,dx \doteq \left(e^{(0.25)^2} + e^{(0.75)^2} + e^{(1.25)^2} + e^{(1.75)^2}\right)(.5) \doteq 14.48561253.$$

(B) (5) Use the Trapezoidal Rule with $n = 4$ to approximate $\int_0^2 e^{x^2}\,dx$.

**Solution:** The Trapezoidal Rule approximation is

$$\int_0^2 e^{x^2}\,dx \doteq \frac{.5}{2}\left(e^{0^2} + 2e^{(0.5)^2} + 2e^{(1.0)^2} + 2e^{(1.5)^2} + e^{(2.0)^2}\right) \doteq 20.64455905.$$

(C) (5) Check the appropriate boxes:

**Solution:** We have $\frac{d}{dx}(e^{x^2}) = 2xe^{x^2}$, and

$$\frac{d^2}{dx^2}(e^{x^2}) = 2e^{x^2} + 4x^2e^{x^2} = (2 + 4x^2)e^{x^2}.$$

Since this is $> 0$ for all real $x$, the midpoint approximation is an underestimate, and the trapezoidal rule approximation is an overestimate, because $e^{x^2}$ is concave up on $[0, 2]$.

*Note:* From this point on in the exam, if an entry from the table of integrals applies, you may use it for full credit if you state which entry you are using and indicate what $u$ and what constant values $a$, $b$, etc. are involved.

IV. For each of the following improper integrals, set up and evaluate the appropriate limits to determine whether the integral converges. If so, find its value; if not, say "does not converge."

(A) (6) $\displaystyle\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx.$

**Solution:** The function $f(x) = \frac{1}{\sqrt[3]{x-1}}$ has an infinite discontinuity at $x = 1$, which is in the middle of the interval $[0, 2]$. Hence we must identify

$$\lim_{b\to 1^-}\int_0^b \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{b\to 1^-}\frac{3}{2}(x-1)^{2/3}\Big|_0^b = \lim_{b\to 1^-}\frac{3}{2}(b-1)^{2/3} - \frac{3}{2} = \frac{-3}{2},$$

and

$$\lim_{a\to 1^+}\int_a^2 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{a\to 1^+}\frac{3}{2}(x-1)^{2/3}\Big|_a^2 = \lim_{a\to 1^+}\frac{3}{2} - \frac{3}{2}(a-1)^{2/3} = \frac{3}{2}.$$

Since both limits exist, the integral converges to $\frac{-3}{2} + \frac{3}{2} = 0$.

(B) (9) $\displaystyle\int_0^\infty \frac{1}{(x^2 + 4x + 5)^{3/2}}\,dx.$

**Solution:** If we complete the square inside the $-3/2$ power, we have $x^2 + 4x + 5 = (x+2)^2 + 1$. Therefore, after the substitution $u = x + 2$, $du = dx$, we have

$$\int_2^\infty \frac{1}{(u^2+1)^{3/2}}\,du = \lim_{b\to\infty} \int_2^b \frac{1}{(u^2+1)^{3/2}}\,du.$$

This is the form of # 29 in the table of integrals with $a = 1$. Hence we must determine if

$$\lim_{b\to\infty} \left.\frac{u}{\sqrt{1+u^2}}\right|_2^b = \lim_{b\to\infty} \frac{b}{\sqrt{1+b^2}} - \frac{2}{\sqrt{5}}$$

exists or not. The first term can be rewritten as

$$\frac{b}{\sqrt{1+b^2}} = \frac{b\cdot\frac{1}{b}}{\sqrt{1+b^2}\cdot\frac{1}{b}} = \frac{1}{\sqrt{\frac{1}{b^2}+1}}.$$

Hence

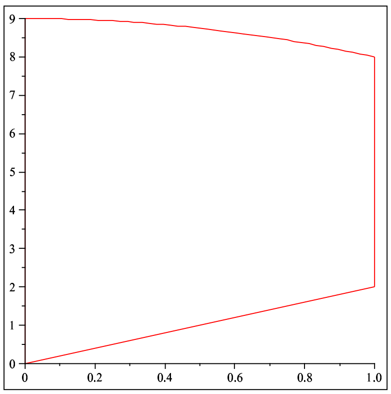
$$\lim_{b\to\infty} \frac{1}{\sqrt{\frac{1}{b^2}+1}} = 1,$$

so the integral converges, to the value $1 - \frac{2}{\sqrt{5}} = \frac{\sqrt{5}-2}{\sqrt{5}}$.

V. A region $R$ in the plane is bounded by the graphs $y = 9 - x^2$, $y = 2x$, $x = 0$ and $x = 1$.

(A) (5) Sketch the region on the following axes:

**Solution:**




(B) (5) Compute the area of the region $R$.

**Solution:** The area is given by the integral

$$A = \int_0^1 9 - x^2 - 2x\,dx = 9x - \frac{x^3}{3} - x^2\Big|_0^1 = \frac{23}{3}.$$

(C) (10) Compute the volume of the solid obtained by rotating $R$ about the $x$-axis.

**Solution:** The cross-sections of the solid by planes perpendicular to the $x$-axis are washers with inner radius $r_{in} = 2x$ and outer radius $r_{out} = 9 - x^2$. So the volume is the integral of the area of the cross-section:

$$\begin{aligned} V = \int_0^1 \pi(9-x^2)^2 - \pi(2x)^2\,dx &= \pi \int_0^1 81 - 22x^2 + x^4\,dx \\ &= \pi\left( 81x - \frac{22x^3}{3} + \frac{x^5}{5}\Big|_0^1 \right) \\ &= \frac{1108\pi}{15}. \end{aligned}$$

(D) (5) Set up the integral(s) to compute the volume of the solid obtained by rotating $R$ about the $y$-axis. *You do not need to compute the value.*

**Solution:** The cross-sections by planes perpendicular to the $y$-axis are all disks, but the function giving the radius of the disk is given by three different formulas depending on whether $0 \le y \le 2$, or $2 \le y \le 8$, or $8 \le y \le 9$.

$$V = \int_0^2 \pi\left(\frac{y}{2}\right)^2 dy + \int_2^8 \pi(1)^2\,dy + \int_8^9 \pi(\sqrt{9-y})^2\,dy.$$

VI. (10) The daily solar radiation $x$ per square meter (in hundreds of calories) in Florida in October has a probability density function $f(x) = \frac{3}{32}(x-2)(6-x)$ if $2 \le x \le 6$, and zero otherwise. Find the *mean* daily solar radiation.

**Solution:** The mean is given by the integral

$$\overline{x} = \int_2^6 x \cdot \frac{3}{32}(x-2)(6-x)\,dx = \int_2^6 \frac{3x^2}{4} - \frac{3x^3}{32} - \frac{9x}{8}\,dx = 4.$$

VII. Does each of the following infinite series converge or diverge? For full credit, you must justify your answer completely by showing how the indicated test applies and leads to your stated conclusion.

(A) (10) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{2n}}$ – Integral Test

**Solution:** The function $f(x) = xe^{-2x}$ is continuous and positive on $[1,\infty)$. Moreover, $f'(x) = e^{-2x}(1-2x) < 0$ for all $x > 1/2$. Therefore $f(x)$ is decreasing on $[1,\infty)$, and the Integral Test applies. The integral can be evaluated by parts with $u = x$, $dv = e^{-2x}$

(this is also the form of #96 in the table of integrals with $a = -2$):

$$\begin{aligned}
\int_1^\infty xe^{-2x}\,dx &= \lim_{b\to\infty} \int_1^b xe^{-2x}\,dx \\
&= \lim_{b\to\infty} \frac{-x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\Big|_1^b \\
&= \lim_{b\to\infty} \frac{-b}{2}e^{-2b} - \frac{1}{4}e^{-2b} + \frac{3}{4e^2} \\
&= \frac{3}{4e^2}.
\end{aligned}$$

Since the integral converges, the series also converges.

(B) (10) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2 e^n}$ – any applicable method.

**Solution:** Apply the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{3^{n+1}}{(n+1)^2 e^{n+1}} \cdot \frac{n^2 e^n}{3^n} \right| = \lim_{n\to\infty} \frac{3}{e}\frac{(n+1)^2}{n^2} = \frac{3}{e} > 1.$$

Therefore the series diverges.

VIII.

(A) (7.5) Using the Ratio Test and testing the endpoints, determine the interval of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n}$.

**Solution:** Applying the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{2^{n+1}x^{n+1}}{n+1} \cdot \frac{n}{2^n x^n} \right| = \lim_{n\to\infty} 2\frac{n}{n+1}|x| = 2|x|.$$

The condition for absolute convergence is $2|x| < 1$, so $|x| < 1/2$. At the endpoints:

- $x = 1/2$: $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, which diverges.
- $x = -1/2$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is an alternating harmonic series which converges by the Alternating Series Test.

Therefore the interval of convergence is $[-1/2, 1/2)$.

(B) (7.5) Find the Taylor series for the function $f(x) = \sin(x/2)$. Express your answer in summation ("sigma") notation.

**Solution:** We have the standard Taylor series for the sine function:

$$\sin(u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!},$$

and that series converges for all real $u$ to $\sin(u)$. Therefore, substituting $u = x/2$ gives

$$\sin(x/2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^{2n+1}(2n+1)!}.$$

IX.

(A) (5) A drug is administered to a patient intravenously at a constant rate of 10mg per hour. The patient's body breaks down the drug and removes it from the bloodstream at a rate proportional to the amount present. Write a differential equation for the function $Q(t)$ = amount of the drug present (in mg) in the bloodstream at time $t$ (in hours) that describes this situation. You *do not need to solve* the equation.

**Solution:** The rate of change of $Q$ is equal to *the rate in minus the rate out*, so the differential equation is

$$\frac{dQ}{dt} = 10 - kQ,$$

for some constant $k$ (the constant of proportionality).

(B) (10) Find the general solution of the differential equation $y' = y\sqrt{e^{2x}+1}$.

**Solution:** This is a separable equation, so we separate variables and integrate to solve for $y$:

$$\begin{aligned}
\int \frac{dy}{y} &= \int \sqrt{e^{2x}+1}\, dx \\
\ln|y| &= \int \frac{\sqrt{u^2+1}}{u}\, du \text{ (letting } u = e^x) \\
\ln|y| &= \sqrt{u^2+1} - \ln\left|\frac{1+\sqrt{u^2+1}}{u}\right| + C \text{ (by \#23 in the table, } a = 1) \\
\ln|y| &= \sqrt{e^{2x}+1} - \ln\left(\frac{1+\sqrt{e^{2x}+1}}{e^x}\right) + C \\
y &= b\frac{e^{x+\sqrt{e^{2x}+1}}}{1+\sqrt{e^{2x}+1}},
\end{aligned}$$

where $b = \pm e^C$.

(C) (10) A population $P$ of tree frogs is undergoing logistic growth following the differential equation $P' = (.1)P(1 - \frac{P}{100})$, $t$ in years. If the initial population is $P(0) = 10$ (in thousands of individuals), how long does it take for the population to reach 45 thousand?

**Solution:** Recall that the general solution of the logistic equation $P' = kP(1-P/M)$ is $P = \frac{M}{1+be^{-kt}}$, where $b$ is an arbitrary constant. (This can be derived by separation of variables, and integration using partial fractions.) Here $M = 100$ and $k = .1$, so

$$P(t) = \frac{100}{1+be^{-(0.1)t}}.$$

If $P(0) = 10$, then

$$10 = \frac{100}{1+b},$$

so $b = 9$. Then we want to solve

$$45 = \frac{100}{1+9e^{-(0.1)t}},$$

so $1 + 9e^{-(0.1)t} = \frac{20}{9}$, and $e^{-(0.1)t} = \frac{11}{81}$. Therefore,

$$t = \frac{\ln(11/81)}{-0.1} \doteq 19.97 \doteq 20 \text{ years} .$$